Entry into complementary good markets with network effects

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Abstract

We examine whether complementarities can help a firm enter a market with strong network effects and incumbency advantages. We provide conditions under which bundling the network good with a complementary good can be an optimal entry strategy, and we further show that this strategy should not be subject to anti-competitive concerns (both in the short and in the long term). When product complementarity is weak enough, we also show that an entrant may prefer a more cooperative approach not based on bundling but rather on extending the complementarity benefits to the incumbent's network good.

Keywords: Network effects, bundling, complementarities, entry.

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1 Introduction

Network effects are widely considered as one of the most difficult entry barriers to overcome (see, *e.g.*, Lieberman and Montgomery, 1998). Google's attempts to become a prominent actor in the social networks market, for example, have been hampered by Facebook and Twitter's incumbency advantages originated in network effects.

A weapon commonly used to counteract this type of entry barrier is to bundle the network good with complementary goods offered in other markets. For example, Microsoft was successful when competing against Netscape in the 1990s because it bundled Internet Explorer with Windows. Bundling allows the entrant to extend its market power from one market to another, and also induces consumers to coordinate on using the entrant's network good to benefit from complementarities. However, bundling also leads to obvious anti-competitive concerns, as in the case of Microsoft vs. Netscape, so its value as a business strategy is dubious.

In contrast to the inherently competitive approach of entry through bundling, other entrants employ a more cooperative approach. This approach is based on enhancing the complementarity between the product originally sold by the entrant and the network good sold by the incumbent. For example, in 2015, Microsoft enabled the interoperability of MS Office and Dropbox, the largely dominant player in the online storage industry. Interestingly, Microsoft's OneDrive competes vigorously with Dropbox in this industry, even nowadays.

The purpose of this paper is to analyze how a firm can capitalize on product complementarity when entering a market with winner-take-all features because of strong enough network effects. In our model, the product originally sold by the entrant is a complement to its network good, but entering the market is difficult because the existing incumbent benefits from optimistic expectations by consumers (Zhu and Iansiti, 2012). We study whether the entrant benefits from using bundled pricing and/or extending complementarity benefits to the incumbent's network good. When the entrant takes actions to extend such benefits to the product sold by the incumbent, as Microsoft did with Dropbox, complementarity is *general*; otherwise, complementarity is *specific*. Specific complementarity typically reflects a lack of interoperability between the entrant's original good and the incumbent's network good, even though it can also reflect cost advantages because of savings in transaction costs by consumers due to one-stop shopping.

Our analysis yields two main results. First, the practice of bundling and that of enhancing product complementarity (interoperability) are substitutes, so they do not reinforce each other: the optimal entry strategies are either unbundled pricing with a general complementarity or bundled pricing with a specific complementarity. This finding is summarized in Figure 1. Intuitively, it reflects the tension between competition and cooperation in achieving the entrant's objectives. As our previous examples show, the competitive (Microsoft-Netscape) and cooperative (Microsoft-Dropbox) approaches are typically used in the real world.

	Unbundled Pricing	Bundling
Specific Complementarity	✗	✓
General Complementarity	✓	✗

Figure 1: Optimal entry strategies: unbundled pricing (bundling) should optimally be associated with a general (specific) complementarity

Second, we show that the desirability of bundling with a specific complementarity relative to unbundled pricing with a general complementarity falls with the strength of network effects, whereas it grows with the degree of complementarity between the entrant's original product and the network goods. Figure 2 illustrates the boundary conditions that determine which one of the two entry strategies is optimal.

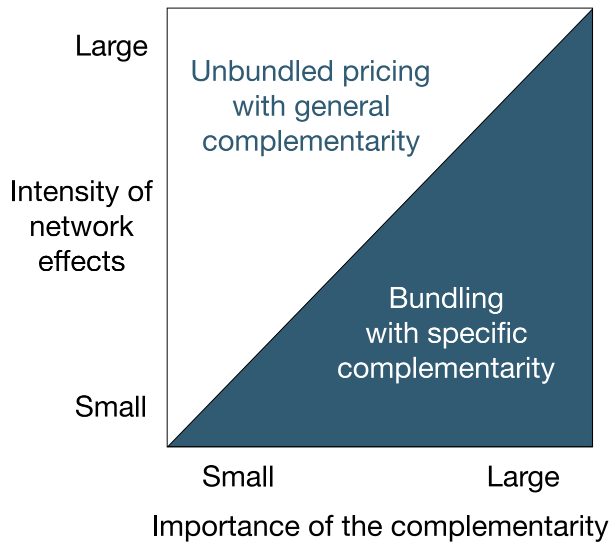

Figure 2: Optimal business model

A key managerial lesson that emerges from the paper is that a high enough degree of specific complementarity coupled with bundling can allow a firm to overcome entry barriers in a profitable way. This may explain why Microsoft was able to overcome the initial network advantages of WordPerfect and Lotus 1-2-3 by bundling its word processor, spreadsheet, and slideshow presentation software together in MS Office.

Another key managerial lesson of the paper is that an entrant may engage in some cooperative activities with a firm that dominates the network market even if doing so significantly hinders entry into such a market. This may explain why Microsoft enhanced the complementarity of Dropbox with MS Office even though it continues to create competition to Dropbox through MS OneDrive, much less consumed than Dropbox and priced separately from MS Office. This apparently foolish behavior by Microsoft in which it collaborates with a highly dominant firm at the same time it keeps its (weaker) product to put price pressure on such a competitor can be rationalized by our theoretical model. The point is that Microsoft can largely appropriate the complementarity gains in the office suite market through appropriate pricing, unlike Dropbox, which cannot raise prices as much in the online storage market because of Microsoft's lower prices in such a market.

A final contribution of our paper is to show that the optimal strategies pursued by the entrant should not raise anti-competitive concerns. This is quite clear for the entry strategy based on unbundled pricing with a general complementarity because of the newly created complementarity benefits. However, an entry strategy based on bundling would typically be considered anti-competitive in the absence of network effects because of the exclusion of the incumbent and the subsequent industry monopolization, even if consumers benefit in the short term. Notwithstanding, when the market is winner-take-all, as in our setting due to the presence of strong network effects, there will be a monopoly in the long term anyway. Entry through bundling allows for the replacement of a "bad monopoly" (the incumbent's) with a "good monopoly" (the entrant's), which increases social welfare.[1] The practices we analyze are therefore shielded against antitrust concerns owing to the complementarity benefits generated by the "good monopoly."

Our paper contributes to the literature dealing with durable market dominance in network industries. This literature has long recognized that network effects constitute significant barriers to entry that may be difficult to overcome by potential entrants (Lieberman and Montgomery, 1988). As Farrell and Klemperer (2007, p. 1972) explain, "consumers' expectations may naturally focus on established firms, so entry with network effects [is] hard." The focus has been mainly on whether an entrant can outcompete an established firm enjoying network effects thanks to technological improvements, as in Farrell and Saloner's (1985) pioneering analysis, or the more recent but also influential paper by Zhu and Iansiti (2012) that emphasizes higher product quality of the entrant. To the best of our knowledge, the only paper explicitly studying entry strategies in network industries is Katz and Shapiro (1992), which focuses on the optimal timing of entry.

Our analysis on optimal entry strategies relies on the relevant but so far neglected roles that product complementarity and bundling may play on the replacement of incumbents in network industries. Our emphasis on these two aspects makes our paper close to

[1] We thank Jacques Crémer for the graphic wording.

seminal work by Eisenmann, Parker and Van Alstyne (2011) on platform envelopment. Our work complements theirs because our theory does not rest upon the entrant capitalizing on network effects in its original market to enter the network market, which is their crucial insight. In addition, we draw conclusions regarding product design decisions and their interaction with bundling and unbundled pricing that have not been explored previously by this literature.

Our link to work on platform envelopment highlights that our paper also contributes to the literature on the strategic and welfare effects of bundling. Bundling was initially treated as a price discrimination scheme that allows for better surplus extraction from heterogeneous consumers, as pointed out by Adams and Yellen (1976), Schmalensee (1984), and McAfee et al. (1989). The modern treatment of bundling has focused however on its market foreclosure aspects. More precisely, the literature has analyzed whether a multi-product monopolist can use bundling to foreclose access of a single-product rival to one of the markets it serves (see Whinston, 1990, Choi, 2001, Carlton and Waldman, 2002, and Nalebuff, 2004).

Our analysis differs from this literature in that we examine the private (and public) implications of bundling when one of the bundled goods is a network good, one of the crucial aspects present in our setting that differentiates our work from previous literature (*e.g.*, Whinston, 1990).[2]

In addition, past research has emphasized a purely competitive approach towards entry into network markets, whereas our integral approach also takes into account the cooperative aspects that underlie making the complementarity general rather than specific (Brandenburger and Nalebuff, 1996). This consideration of a wider range of entry strategies allows us to draw precise managerial prescriptions about the desirability of each *vis-à-vis* others. Finally, we highlight that, contrary to the message from the litera-

[2] For example, Whinston (1990) shows that a firm prefers to be inactive when its competitor uses bundling. In our setting with network effects favoring the incumbent, bundling need not be enough for the entrant to force the incumbent out: the entrant should rely as well on strong enough complementarity (especially when network effects are very intense).

ture on exclusionary bundling, the practices we consider are not anti-competitive given the winner-take-all nature of the network market and the social benefits that product complementarity brings about.

2 The model

We consider a game played by two firms labeled 1 and 2 and a continuum of consumers with unit mass. Consumers demand products in two markets labeled A and B. Firm 1 sells product a in market A and product b_1 in market B, whereas firm 2 sells product b_2 in market B. There are no fixed costs of operation for any of the firms. The marginal costs of production of good a are normalized to zero (which means that the price of such a good should be interpreted as a markup), whereas there is a marginal cost $c \geq 0$ of producing b_i, $i \in \{1, 2\}$.

Consumers are willing to consume at most one unit of the product sold in each of the two markets, and they are identical except for their valuations of product a. In particular, the valuation v of a given consumer is an independent draw from a random variable uniformly distributed between 0 and 1. Each consumer privately observes her valuation before buying any good.

Consumers obtain a gross utility u from consuming any of the goods sold in market B, where u is publicly known. The goods sold in market B exhibit direct network effects. In particular, if $n_i^e \in [0, 1]$ consumers are expected to consume b_i, the expected valuation of a consumer contemplating to purchase such a product increases by $\alpha\, n_i^e$, where α is a known parameter that represents the intensity of network effects.[3] We assume u is large enough, so that market B is fully covered, and $\alpha \geq 1/2$, which, as we will show, implies a winner-take-all outcome for market B.

Products a and b_1 are complementary: a consumer who consumes both a and b_1 in-

[3]We are assuming that networks are incompatible, so the value of product b_i for consumers depends upon the number of users with whom they can interact by using such product (Farrell and Saloner, 1985; Katz and Shapiro, 1985, Economides, 1996).

creases her utility by a fixed amount $\beta \in [0,1)$. The relationship between products a and b_2, on the other hand, depends on firm 1's product design (interoperability) decisions. If firm 1 does not extend complementarity benefits to firm 2, then the increase in utility is not available for a consumer who consumes a and b_2, and we say that the complementarity is firm specific. If firm 1 extends complementarity benefits to firm 2, so that consumers of products a and b_2 increase their utility by β as well, we say that the complementarity is general.[4]

We study two revenue models: unbundled pricing and bundling. With unbundled pricing, firm 1 sets prices p_a for a unit of good a and p_1 for a unit of b_1, and firm 2 sets price p_2 for a unit of b_2. The utility of a consumer who consumes good b_i but not a is

$$U_{b_i} = u + \alpha\, n_i^e - p_i, \tag{1}$$

the utility generated by consuming a and b_1 is

$$U_{ab_1}(v) = v - p_a + \beta + u + \alpha n_1^e - p_1, \tag{2}$$

and the utility generated by consuming a and b_2 is

$$U_{ab_2}(v) = v - p_a + \beta\, G + u + \alpha n_2^e - p_2, \tag{3}$$

where G is an indicator function that takes value 0 if the complementarity is specific and 1 if it is general.

Under bundling, firm 1 chooses price p for a bundle composed by one unit of a and one unit of b_1, and firm 2 sets price p_2 for a unit of b_2.[5] We assume that consumers can freely dispose of product b_1 if they buy the bundle from firm 1 and are interested in

[4]The case of $\beta \geq 1$ yields no additional insights and unnecessarily lengthens proofs and propositions. Results are available upon request.

[5]We thus focus on pure bundling. In the online appendix, we examine the implications of having firm 1 charge a price in each of the markets in which it operates together with a discount offered to those consumers who purchase both of its products (mixed bundling). We show that there is no loss of generality in restricting our analysis to pure bundling, a result that parallels the one obtained by Whinston (1990) in a setting without network effects.

consuming b_2 rather than b_1. The utility generated by consuming a and b_1 is

$$U_{ab_1}(v) = v - p + \beta + u + \alpha n_1^e, \tag{4}$$

the utility generated by consuming a and b_2 is

$$U_{ab_2}(v) = v - p + \beta G + u + \alpha n_2^e - p_2. \tag{5}$$

and the utility generated by consuming only b_2 is

$$U_{b_2} = u + \alpha n_2^e - p_2. \tag{6}$$

We consider a three-period game. In the first period, firm 1 decides whether to make the complementarity specific or general, and whether to use bundled or unbundled pricing. Therefore, firm 1 chooses among four business models: unbundled pricing with a specific complementarity (US), unbundled pricing with a general complementarity (UG), bundling with a specific complementarity (BS), and bundling with a general complementarity (BG). In the second period, firms set prices. In the third period, having observed all prices, consumers form rational expectations and simultaneously decide which goods to consume. Our solution concept is subgame perfect equilibrium.

In solving for a subgame perfect equilibrium, we work backwards and begin by considering the third period. Given a set of prices, it will usually be the case that there exist several Nash equilibria for the adoption game played by consumers. Given that we are interested in contemplating firm 1 as an entrant, we will always select the second-period Nash equilibrium with the largest market share for firm 2. Thus, when pricing is such that consumers can coordinate in several ways, consumers believe that the prevailing equilibrium is the one favoring firm 2. This refinement is meant to capture historical inertia favoring the incumbent's product (*e.g.*, existence of an installed base of users), and its main implication is to handicap entry by firm 1. Such pessimistic beliefs for the entrant are a standard way to represent incumbency advantages in static models with network effects (see, for example, Griva and Vettas, 2011; and Caillaud and Jullien, 2003). The promising developments by Biglaiser and Crémer (2016) and Halaburda et al. (2016) provide truly dynamic models with a similar flavor.

It is worth noting that firms in our setting are assumed to be symmetric except for two aspects. On the one hand, firm 1 is active in the market of a complementary good. On the other hand, firm 2 benefits from optimistic expectations by consumers. As a consequence, the higher the intensity of network effects, captured by parameter α, the more difficult is for firm 1 to enter.[6]

3 Equilibrium market interaction

We start by solving the second and third periods of the game, *i.e.* we characterize the equilibrium prices and profits that result from each of the four business models introduced above. We start with unbundled pricing with a specific complementarity (US). Next, we consider the case in which the entrant keeps separate prices but decides to share its complementarity with the incumbent (UG). We then examine the consequences of having firm 1 charge a single price for the bundle of the two products it sells, for the cases of a specific (BS) and a general complementarity (BG).

3.1 Unbundled pricing with specific complementarity (US)

Consider utility expressions (1)-(3) with $G = 0$. Given p_a, p_1 and p_2, let $n_i^e \in [0, 1]$ denote the number of consumers who are expected to purchase b_i in market B. Letting $n_i \in [0, 1]$ denote the number of consumers who actually purchase b_i in market B, the fact that each consumer is negligible and forms rational expectations implies that, given prices, $n_i^e = n_i$ for all $i \in \{1, 2\}$. As market B is assumed to be covered, it follows that $n_1 = 1 - n_2$, so we simply need to find out how much demand is generated by firm 2 in market B given the

[6]If such inertia favored entrants, then network benefits would favor entry rather than hinder it.

prices charged by both firms.

Any pricing strategy for firm 1 that involves $p_a > 1 + \beta$ is (weakly) dominated by a strategy with $p_a = 1 + \beta$. Indeed, given our assumption on v, no consumer would buy a and b_1 when $p_a > 1 + \beta$, as buying b_1 in isolation would entail a higher level of utility. Similarly, firm 1 cannot be playing any pricing strategy that involves $p_a < 0$ because it is (weakly) dominated by a strategy with $p_a = 0$. Thus, in what follows we consider strategies such that $p_a \in [0, 1 + \beta]$. It is also worth noting that, because consumers can freely dispose of product b_1, it must hold that firm 1 finds it optimal to charge $p_1 \geq 0$.

In the proof of Lemma 1 below, we derive firm 2's demand correspondence for all admissible values of p_2, p_1 and p_a. Such a correspondence is graphically represented in the left panels of Figure 3, in which we distinguish three cases, depending on the value of p_a. Recall that there is no need to plot firm 1's demand correspondence because its demand equals $1 - n_2$.

Given that in case of multiplicity we select the third-period Nash equilibrium most favorable to firm 2, the right panels of Figure 3 represent firm 2's demand function. For example, when $p_1 + p_a - \alpha - \beta < p_2 < p_1 + \alpha - \beta$ in Figure 3's left top panel, our refinement implies that firm 2 must necessarily be capturing the whole network market, as one can see on the corresponding right top panel. Indeed, only when $p_2 > p_1 + \alpha - \beta$ does firm 2 fail to make some sales in this market, all of which are made by firm 1.

The following lemma shows the equilibrium of the second and third periods of the game (results not proven in the text can be found in the Appendix).

Lemma 1 (Unbundled pricing with a specific complementarity). *An equilibrium exists and is unique. The equilibrium is such that:*

(a) If $\alpha < \min\{\beta(6 + \beta)/4, c + \beta\}$, *then firm 1 captures the network goods market B* ($n_2^{US} = 0$), $p_a^{US} = (1 + \beta)/2$, $p_1^{US} = c + \beta - \alpha$, $p_2^{US} = c$, $\pi_1^{US} = \beta - \alpha + (1 + \beta)^2/4 > 0$, *and* $\pi_2^{US} = 0$. *Firm 1 uses good* b_1 *as a loss leader*[7] *in equilibrium* ($p_1^{US} < c$) *if and only if* $\beta < \alpha$.

[7] A product is a loss leader if each unit is sold below cost in order to stimulate sales of a complementary

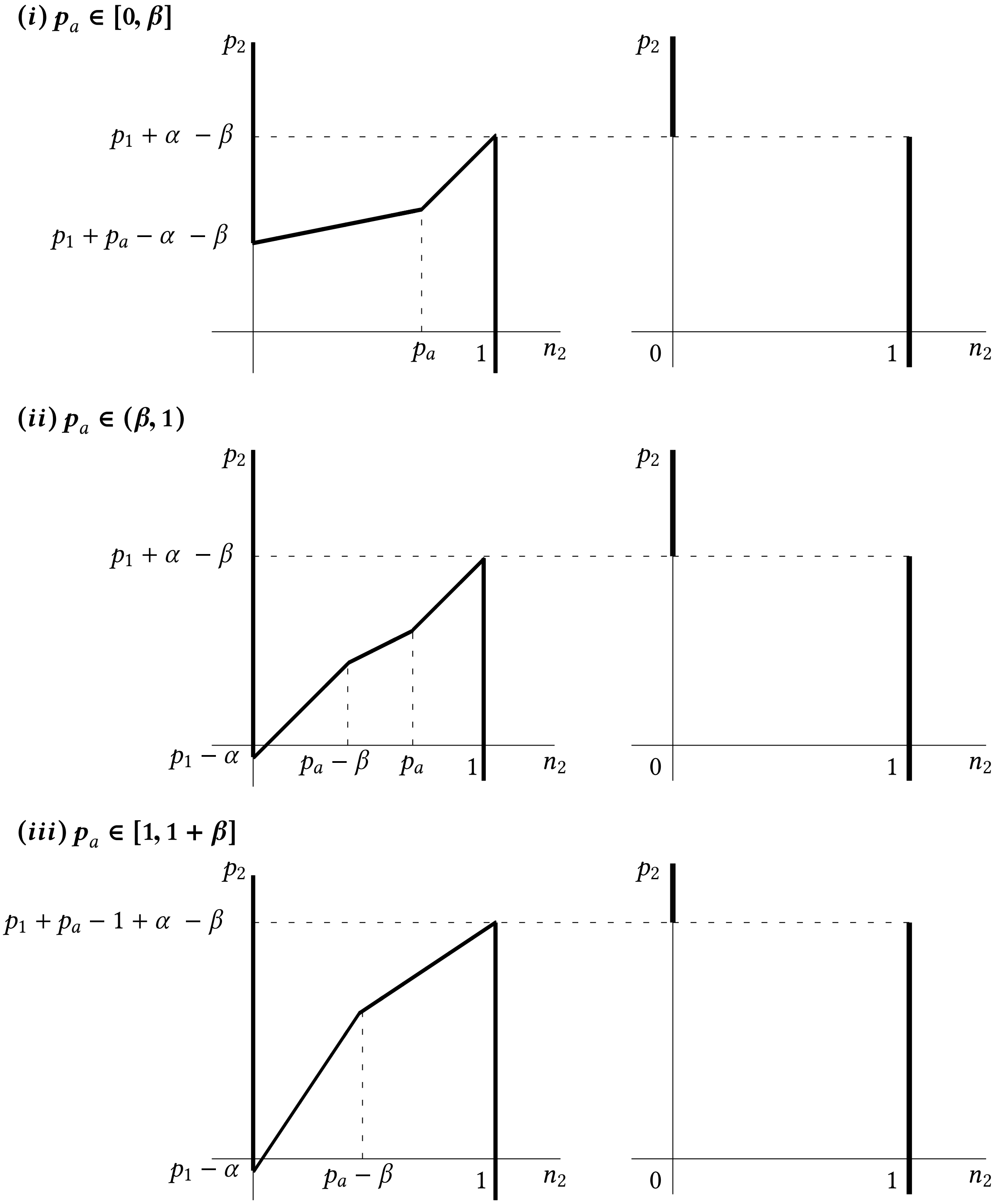

Figure 3: Firm 2's demand correspondence (left panels) and demand function with the equilibrium refinement (right panels) for different values of p_a

(b) If $\alpha \geq \min\{\beta(6+\beta)/4, c+\beta\}$, then firm 2 maintains its dominance of the network goods market B ($n_2^{US} = 1$), $p_a^{US} = 1/2$ and $\pi_1^{US} = 1/4$. Firm 1 uses good b_1 as a loss leader, but has no sales of this good in equilibrium. In addition:

(b1) If $c < \beta(2+\beta)/4$, then $p_1^{US} = 0$, $p_2^{US} = \alpha - \beta$, and $\pi_2^{US} = \alpha - \beta - c \geq 0$.

(b2) If $c \geq \beta(2+\beta)/4$, then $p_1^{US} = c - \beta(2+\beta)/4 \geq 0$, $p_2^{US} = c + \alpha - \beta(6+\beta)/4$, and $\pi_2^{US} = \alpha - \beta(6+\beta)/4 > 0$.

Lemma 1 shows that firm 1's equilibrium profit (weakly) increases with β and (weakly) decreases with α, whereas the converse holds for firm 2. This result is a direct consequence of firm 1 exclusively enjoying the complementarity benefit and of firm 2 exclusively enjoying the incumbency advantage with regards to network effects.

The top panels of Figure 4 represent the subset of parameters for each type of equilibrium in Lemma 1.[8] As it can be readily seen in the figure, for given values of α and c, there exists a threshold level of β such that the equilibrium has $n_2^{US} = 1$ for values of β below the threshold, and $n_2^{US} = 0$ for values of β above the threshold. This threshold increases with α, which is natural given firm 2's incumbency advantage, and shifts inwards as c grows, which is also natural because c affects firm 2 more than firm 1, given that the latter is active in market A as well as in market B.

Part (a) of Lemma 1 shows that, given α and c, firm 1 enters the market of the network good if β is large enough, an intuitive result. Perhaps more surprising is the mechanism at play, because the complementarity benefit β plays a dual role: (i) it enables firm 1 to price firm 2 out of market B; and (ii) it also enables firm 1 to raise its price in market A because of the complementarity between a and b_1. Indeed, the latter role interacts

good. Under competition, a multi-product firm may engage in loss leading not only with that purpose but also to outcompete a single-product rival and exclude it from the market.

[8]Based on our assumption on α, in the vertical axis of Figure 4 we consider $\alpha \geq 1/2$. As a consequence, $\beta(6+\beta)/4 > 1/2$ when $\beta > \sqrt{11} - 3 \simeq 0.33$. Note that $\beta(6+\beta)/4 > c + \beta$ when $\beta > \sqrt{4c+1} - 1$ (or $c < \beta(\beta+2)/4$, as reported in Lemma 1), and vice versa. As $\alpha \geq 1/2$ and $\beta \in [0,1)$ by assumption, it holds that $\sqrt{4c+1} - 1 \geq 1$ if $c \geq 3/4$ and $\sqrt{4c+1} - 1 > \sqrt{11} - 3$ if $c \geq 7/2 - \sqrt{11} \simeq 0.18$. It follows that $c + \beta < \beta(6+\beta)/4$ when $c \in (0, 0.18]$, whereas $c + \beta \geq \beta(6+\beta)/4$ when $c \geq 3/4$. These cases are respectively represented in the left and right panels of Figure 4. When $c \in (0.18, 3/4)$, the two threshold values of α intersect in $\beta = \sqrt{4c+1} - 1$, as it can be seen in the central panels of Figure 4.

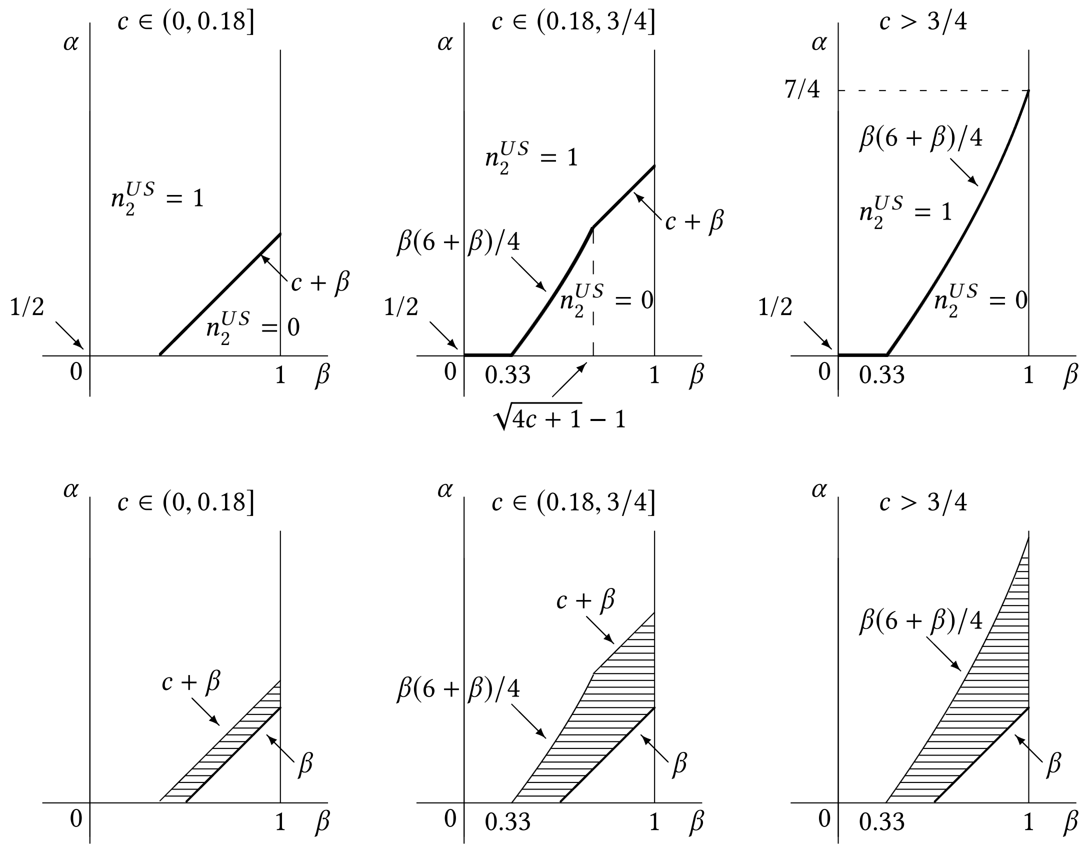

Figure 4: Equilibrium market shares (top panels) and successful entry with loss leading (shaded area in bottom panels)

with the former: the strong position of firm 1 in market A induces it to compete more fiercely in market B so as to exploit the complementarity benefit, thereby using b_1 as a loss leader to expand good a's market. In fact, $p_1^{US} - c = \beta - \alpha$ is negative when β is slightly larger than the threshold value that delineates whether firm 1 conquers market B. As the complementarity benefit β grows, firm 1 can increase b_1's markup, and, when it exceeds α, b_1's markup becomes positive: its advantage over firm 2 is so strong that it does not need to incur a cost to make its monopolistic position in market A more valuable. The shaded areas in the bottom panels of Figure 4 represent the parametric values for which firm 1 uses b_1 as a loss leader and captures the network goods market (given c, these are the values of β and α such that $\beta \leq \alpha < \min\{\beta(6+\beta)/4, c+\beta\}$).

Part (b) of Lemma 1 shows that, given α and c, firm 2 maintains its dominance of market B when β is sufficiently small. In this case, firm 1 is induced to price b_1 at its perceived marginal cost, which is its actual marginal cost c in market B minus the profit increase in market A associated with conquering market B, namely $(1+\beta)^2/4 - 1/4 = \beta(2+\beta)/4$. This last component captures firm 1's willingness to engage in loss leading in market B. On the one hand, when such component is more prominent than c (*i.e.*, $c < \beta(2+\beta)/4$, case (b1)), the fact that p_1^{US} cannot be negative because of free disposal implies that $p_1^{US} = 0$. As a result, firm 2's price always falls with β because those consumers purchasing firm 1's good in market A also weigh whether to purchase the other good sold by firm 1 or to acquire that of firm 2. Because this comparison depends on β, and greater values of β makes firm 2's good relatively less appealing, firm 2 is led to decrease its price as β grows. On the other hand, when $c \geq \beta(2+\beta)/4$ (case (b2)), it holds that $p_1^{US} = c - \beta(2+\beta)/4 \geq 0$, so firm 2's price falls with β for the same reason as in case (b1) as well as because firm 1's incentive to engage in loss leading in market B becomes more intense as β grows. Even though such a loss leading strategy is ineffective, it constraints the equilibrium price of firm 2, which decreases with the degree of complementarity. Note that in case (b) firm 1 could use a more aggressive loss leading approach to conquer market B, but it refrains from doing so because the cost would be too high in comparison with the market expansion benefits that it brings.

A final aspect worth highlighting regarding Lemma 1 is that it shows that firm 1 has no incentives to use good a as a loss leader. The point is that lowering p_a does not expand the demand for good b_1, as can be easily seen in Figure 3.

3.2 Unbundled pricing with a general complementarity (UG)

In the case of unbundled pricing with a general complementarity, utility expressions are still given by (1)-(3), but now $G = 1$. In the Appendix (proof of Lemma 2 below) we show both how to construct and how to represent firm 2's demand correspondence. Figure 5 displays firm 2's demand function once the third-period equilibrium refinement is used.

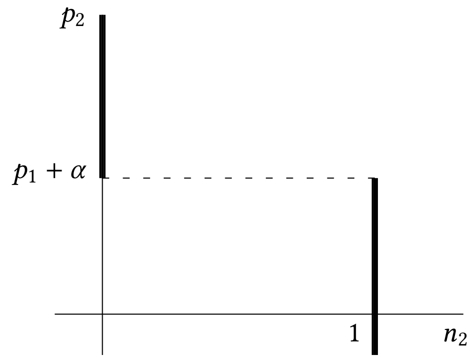

Figure 5: Firm 2's demand function in the unbundled pricing game with a general complementarity

Let us first try to sustain $n_2 = 1$ in equilibrium. Clearly, firm 1 must be charging the monopoly price in market A given that customers' utilities increase by β, so

$$p_a = \underset{0 \leq \widehat{p}_a \leq 1+\beta}{\arg\max} \left\{ \widehat{p}_a \, \min\{1, 1+\beta-\widehat{p}_a\} \right\} = (1+\beta)/2, \tag{7}$$

and it must be earning

$$\pi_1 = (1+\beta)^2/4. \tag{8}$$

Also, firm 2 must be charging the highest possible price such that its demand equals 1. If firm 1 is charging p_1 in market B, then firm 2 must be charging $p_2 = p_1 + \alpha$ (see Figure 5). Firm 1's optimal deviation is $\widehat{p}_1 = p_1 - \epsilon$ (for $\epsilon > 0$ arbitrarily small) so as to attract all consumers in market B, keeping at the same time the price charged in market A as fixed. In order for such a deviation not to be profitable, we must have $p_1 = c$, which implies that $p_2 = c + \alpha$. A similar argument can be used to show that there cannot be an equilibrium in which $n_2 = 0$.

Lemma 2 (Unbundled pricing with a general complementarity)**.** *An equilibrium exists and is unique. Firm 2 maintains its dominance in the network goods market B ($n_2^{UG} = 1$), with $p_a^{UG} = (1+\beta)/2$, $p_1^{UG} = c$ and $p_2^{UG} = c + \alpha$. Firm 1 gains $\pi_1^{UG} = (1+\beta)^2/4$, and firm 2 gains $\pi_2^{UG} = \alpha$.*

Relative to Lemma 1, competition in market B is relaxed and firm 1 can always incorporate the complementarity benefit into its market A price-setting, even though it cannot hope to conquer market B because of firm 2's incumbency advantage.

A natural question to ask is what happens if the complementarity we have just analyzed arises because customers have access to any network regardless of which good in market B they consume. In case of such compatibility between the networks of firms 1 and 2, someone who buys either b_1 or b_2 expects to increase her utility by $\alpha(n_1^e + n_2^e) = \alpha$ and purchasing good a in addition further increases her utility by β. This situation corresponds to suppressing the network effect (*i.e.*, setting $\alpha = 0$) in the above analysis.[9] Therefore, compatibility harms firm 2 without bringing an extra profit to firm 1. If firm 2 could refuse it, and doing so also implied forgoing enjoying the complementarity benefit β, it would hold that firm 2 would not hesitate in remaining incompatible and thus keep its incumbency advantage. Even though this kind of compatibility offer by firm 1 would be a poison pill for firm 2, Lemmas 1 and 2 imply that firm 2 would be happy to instead accept an offer to enjoy the complementarity benefit β.

3.3 Bundling with a specific complementarity (BS)

Consider now utility expressions (4)-(6) with $G = 0$. In the Appendix (proof of Lemma 3 below), we show how to find out firm 2's demand correspondence, together with its graphical representation. Figure 6 represents the demand function that results from using the equilibrium refinement described in Section 2.

The most noteworthy feature of the demand function illustrated in Figure 6 is that firm 2's sales do not vary with p if $p \leq 1$. On the one hand, if $p > 1$, it holds that $U_{ab_2}(v) < U_{b_2}$ for all v, so consumers do not consider buying b_2 together with firm 1's bundled goods. Thus, the relevant utility comparison for a consumer with valuation v for good a is U_{b_2} versus $U_{ab_1}(v)$, a comparison that depends on p. Consequently, firm 2's

[9]Note that the assumption that $\alpha \geq 1/2$ was actually not used in any part of the analysis we performed, so one can set $\alpha = 0$ directly.

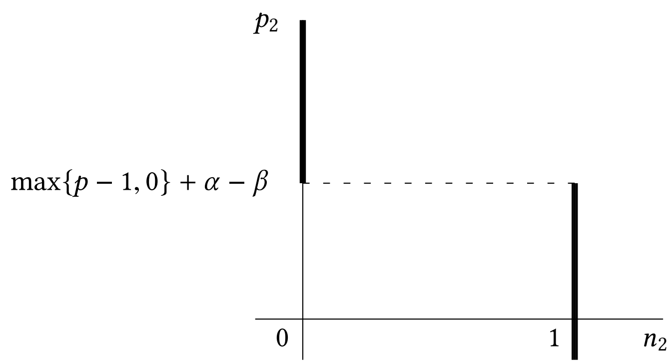

Figure 6: Demand function with bundling and a specific complementarity

demand depends on p for $p > 1$. If $p \leq 1$, on the other hand, consumption of b_2 alone does not dominate consumption of a and b_2 for all v, and the comparison between $U_{ab_1}(v)$ and $U_{ab_2}(v)$ does not depend on p. Because of firm 2's incumbency advantage, consumers choose to coordinate on $n_2 = 0$ only if $p_2 > \alpha - \beta$, and they choose to coordinate on $n_2 = 1$ otherwise. Thus, firm 2's demand does not depend on p if $p \leq 1$.

It is straightforward to characterize equilibria with the aid of Figure 6. When $\alpha \geq c + \beta$, firm 2 can always guarantee conquering market B by charging price $p_2 = \alpha - \beta \geq c$. Provided $p < 1$, firm 2 has no incentive to deviate, whereas firm 1 best responds to such pricing by focusing on market A and charging the monopoly price in this market given marginal cost c, so $p = (1 + c)/2 < 1$ as long as $c < 1$. Therefore, a unique equilibrium exists when $\alpha \geq c + \beta$ and $c < 1$, and it is such that $n_2 = 1$, with profits for firms 1 and 2 equal to $\pi_1 = (1 - c)^2/4$ and $\pi_2 = \alpha - \beta - c$.

It remains to study whether we can have an equilibrium with $n_2 = 1$ and $p \geq 1$. In such a case, firm 2 should be charging price $p_2 = \alpha - \beta + p - 1$. Firm 1 should have no incentive to slightly reduce p and get all the demand. Thus, it should hold that $p - \epsilon - c < 0$ for any arbitrarily small $\epsilon > 0$, so we should have $p = c$ and $p_2 = \alpha - \beta + c - 1$, with $1 \leq p = c$ and $\alpha \geq 1 + \beta$ in order to have an equilibrium such that $n_2 = 1$ and $p \geq 1$.

By an analogous argument, firm 2 cannot profitably capture market B when $\alpha < \min\{1, c\} + \beta$. Indeed, firm 2 must be optimally charging $p_2 = c$ if such an equilibrium exists. Given this pricing, firm 1 must be charging some $p \leq 1+\beta-\alpha+c$ so that it serves all consumers (in both markets). It follows that $p = 1 + \beta - \alpha + c \geq 1$ in order for firm 1 to maximize profit. Because $\alpha < \min\{1, c\} + \beta$, we have that the unique equilibrium is such that $n_2 = 0$, with profits for firms 1 and 2 equal to $\pi_1 = 1 + \beta - \alpha \geq 0$ and $\pi_2 = 0$.

The following lemma summarizes these findings.

Lemma 3 (Bundling with a specific complementarity). *An equilibrium exists and is unique. The equilibrium is such that:*

(a) If $\alpha < \min\{1, c\} + \beta$*, then firm 1 captures the network goods market B (*$n_2^{BS} = 0$*),* $p^{BS} = 1 + \beta - \alpha + c$*,* $p_2^{BS} = c$*,* $\pi_1^{BS} = 1 + \beta - \alpha$*, and* $\pi_2^{BS} = 0$*. In equilibrium, no consumer who buys the bundle disposes of good* b_1*.*

(b) If $\alpha \geq \min\{1, c\} + \beta$*, then firm 2 maintains its dominance of the network goods market B (*$n_2^{BS} = 1$*),* $p^{BS} = \max\{c, (1+c)/2\}$*,* $p_2^{BS} = \alpha-\beta+c-\min\{1, c\}$*,* $\pi_1^{BS} = (\max\{0, (1-c)/2\})^2$*, and* $\pi_2^{BS} = \alpha - \beta - \min\{1, c\}$*. In equilibrium, all consumers who buy the bundle dispose of good* b_1*.*

All consumers purchase some good in market B. When $\alpha < \min\{1, c\} + \beta$, the fact that good b_2 is (rationally) expected to yield no network benefits implies that all consumers want to acquire good b_1 even if b_2 is sold at its marginal cost. Because acquiring good b_1 requires purchasing a as well, bundling implies that firm 1 is able to effectively capture all demand when the complementarity benefit β is large enough to offset firm 2's incumbency advantage, captured by α. As in Lemma 1, the greater α is, the greater the complementarity benefit needs to be in order for firm 1 to conquer the network market.

3.4 Bundling with a general complementarity (BG)

Consider again utility expressions (4)-(6) but with $G = 1$ due to general complementarity. In the proof of Lemma 4, we derive firm 2's demand correspondence. Figure 7 shows the demand function that results from using the equilibrium refinement.

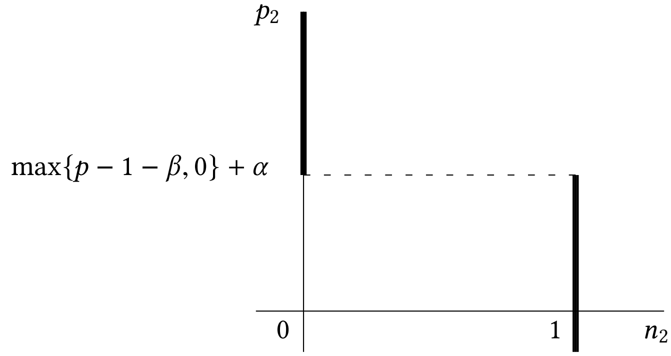

Figure 7: Demand function with bundling and general complementarity

It is straightforward to characterize equilibria with the aid of Figure 7. When $\alpha \geq c$, firm 2 can always guarantee conquering market B by charging price $p_2 = \alpha \geq c$. Provided $p \leq 1 + \beta$, firm 2 has no incentive to deviate, whereas firm 1 best responds to such pricing by focusing on market A and charging the monopoly price in such a market given marginal cost c, so $p = (1 + \beta + c)/2 \leq 1 + \beta$ as long as $c \leq 1 + \beta$. Therefore, a unique equilibrium exists when $\alpha \geq c$ and $c \leq 1 + \beta$, and it is such that $n_2 = 1$, with profits for firms 1 and 2 equal to $\pi_1 = (1 + \beta - c)^2/4$ and $\pi_2 = \alpha - c$. Note that firm 1 makes no sales if it charges a price above $1 + \beta$. Hence, in order for an equilibrium with $n_2 = 1$ and $p \geq 1 + \beta$ to exist, we need that $p = c \geq 1 + \beta$, so $p_2 = \alpha + c - 1 - \beta$, with profits for firms 1 and 2 equal to $\pi_1 = 0$ and $\pi_2 = \alpha - 1 - \beta$. In order for firm 2 not to have an incentive to deviate, we also need $\alpha \geq 1 + \beta$.

As for an equilibrium in which firm 2 does not capture market B, it is clear from Figure 7 that any $p < 1 + \beta$ is dominated by $p = 1 + \beta$, so let us seek for an equilibrium with $p \geq 1 + \beta$. In order for firm 2 not have an incentive to deviate, we need $p - 1 - \beta + \alpha \leq c$. Because firm 1 sells the bundle to all consumers, it earns $p - c$, so $p - c \leq 1 + \beta - \alpha$ implies that $\alpha \leq 1 + \beta$ must be met in order for firm 1 not to earn a negative margin. Because firm 1 maximizes profit by setting $p = 1 + \beta - \alpha + c$ in such a case, $p \geq 1 + \beta$ implies that $\alpha \leq c$ is also required for an equilibrium with $n_2 = 0$ to exist. In summary, it holds when $\alpha \leq c$ and $\alpha \leq 1 + \beta$ that $p = 1 + \beta - \alpha + c$ and $p_2 = c$ is the unique equilibrium such that

$n_2 = 0$, with $\pi_1 = 1 + \beta - \alpha$ and $\pi_2 = 0$. The following lemma summarizes these findings.

Lemma 4 (Bundling with a general complementarity)**.** *An equilibrium exists and is unique. The equilibrium is such that:*

(a) If $\alpha < \min\{c, 1 + \beta\}$, then firm 1 captures the network goods market B ($n_2^{BG} = 0$), $p^{BG} = 1 + \beta - \alpha + c$, $p_2^{BG} = c$, $\pi_1^{BG} = 1 + \beta - \alpha$, and $\pi_2^{BG} = 0$. In equilibrium, no consumer who buys the bundle disposes of good b_1.

(b) If $\alpha \geq \min\{c, 1 + \beta\}$, then firm 2 maintains its dominance of the network goods market B ($n_2^{BG} = 1$), $p^{BG} = \max\{c, (1 + \beta + c)/2\}$, $p_2^{BG} = \alpha + c - \min\{1 + \beta, c\}$, $\pi_1^{BG} = (\max\{0, (1 + \beta - c)/2\})^2$, $\pi_2^{BG} = \alpha - \min\{1 + \beta, c\}$. In equilibrium, all consumers who buy the bundle dispose of good b_1.

In contrast with the case of unbundled pricing, in which entry was not possible with a general complementarity, bundling allows firm 1 to capture the network goods market even if it shares its complementarity. With a general complementarity, if firm 1 opts for unbundled pricing it completely renounces to its advantage when competing with firm 2, hence it cannot hope to conquer market B. With bundling, firm 1 can still capture the network goods market, because those consumers interested in product a obtain product b_1 as part of the bundle, which gives firm 1 a competitive edge against firm 2.

4 Optimal product design and pricing strategies

In this section, we study the solution to the first period of the game, that is, we study both the product design and the pricing strategy of firm 1 that maximize its profit.

We proceed to build intuition by discussing a series of intermediate results, and finish our discussion with Proposition 1, the most important result of the paper.

Lemma 5. *Conditional on capturing market B, the entrant prefers bundling.*

Lemma 5 holds trivially if the complementarity is general, because in this case it is impossible for the entrant to capture market B with unbundled pricing (Lemma 2).

Conditional on capturing market B, if the complementarity is specific and the entrant chooses unbundled pricing, it obtains $\frac{(1+\beta)^2}{4} + \beta - \alpha$ (Lemma 1). If the entrant chooses bundling instead, it captures $1 + \beta - \alpha$ (Lemma 3), which is always larger.[10]

The following result follows from respectively comparing Lemma 1 and Lemma 2 with Lemma 3 and Lemma 4.

Lemma 6. *Conditional on not capturing market B, the entrant prefers unbundled pricing.*

The rationale for this is that, if firm 1 fails to capture market B (and hence fails to increase revenues relative to unbundled pricing), then bundling implies a redundant cost, since the bundle includes a unit of good b_1, which is costly to produce. Thus, firm 1 will never choose bundling if it does not allow it to capture market B.

Our next result relates bundling as an optimal entry strategy with product design.

Lemma 7. *If bundling is optimal, the entrant prefers to have a specific complementarity.*

By Lemma 6, if bundling is optimal, then it must be because firm 1 is better off conquering market B. But if capturing market B is optimal, then there is no advantage in making the complementarity general, given that in this case firm 1 prefers to prevent firm 2 from enjoying complementarity benefits.

Our next result now relates unbundled pricing as an optimal entry strategy with product design.

Lemma 8. *If unbundled pricing is optimal, the entrant prefers to have a general complementarity.*

By Lemma 5, if unbundled pricing is optimal, firm 1 is better off not conquering market B. But if capturing market B is not optimal, then firm 1 benefits from making the complementarity general, in order to increase the valuation of consumers for product a: given that in equilibrium all consumers buy product b_2, their marginal valuation for

[10] We have assumed $\beta < 1$ for simplicity. For $\beta \geq 1$, optimal prices with unbundled pricing are at a corner solution, and bundling and unbundled pricing lead to the same profits.

product a is v if the complementarity is specific, and $v + \beta$ if the complementarity is general.

Together, Lemmas 7 and 8 imply that bundling should optimally be associated with a specific complementarity, and unbundled pricing should optimally be associated with a general complementarity. The next proposition shows the optimal business model as a function of the fundamental parameters of the model, α and β.

Proposition 1 (Optimal business model)**.** *The entrant prefers bundling with a specific complementarity if* $\alpha \leq \min\{(3-\beta)(1+\beta)/4, c+\beta\}$*, and prefers unbundled pricing with a general complementarity otherwise.*

Proposition 1 follows from the previous analysis: bundling is optimal only if it leads firm 1 to capture market B. But the profits from capturing market B depend negatively on α: given the incumbency advantage, network effects are a hurdle that must be overcome to enter the market. Thus, the larger α, the less desirable it is to choose bundling.

Explaining the effect of β on the optimal business model is slightly more difficult, because both the profit from models BS and UG increase with β. However, it is easy to show that increasing β increases the incentives to use bundling.

The reason is that with bundling, the entrant captures the entire surplus created by an increase in β, whereas in the case of unbundled pricing it captures only part of this surplus. Formally, from Lemmas 3 and 4, it follows that the difference between the profits of the two business models is $\pi^{BS} - \pi^{UG} = 1 + \beta - \alpha - \frac{(1+\beta)^2}{4}$, which is increasing in β.

Figure 8 shows the optimal business model as a function of the intensity of network effects (α) and the importance of the complementarity for consumers (β). Notice that the region where bundling with a specific complementarity is preferred enlarges with c. Even though changes in c affect neither π_1^{UG} nor π_1^{BS} (when market B is captured by firm 1), the result is driven by the change in the threshold found in Lemma 3. Greater c makes it harder for firm 2 to keep the network market when firm 1 uses business model BS because firm 1 does not fully pass through the increase in c to consumers, unlike firm

2; hence, the latter is more affected by the greater c, and capturing market B becomes easier through bundling, which makes it more attractive relative to business model UG.

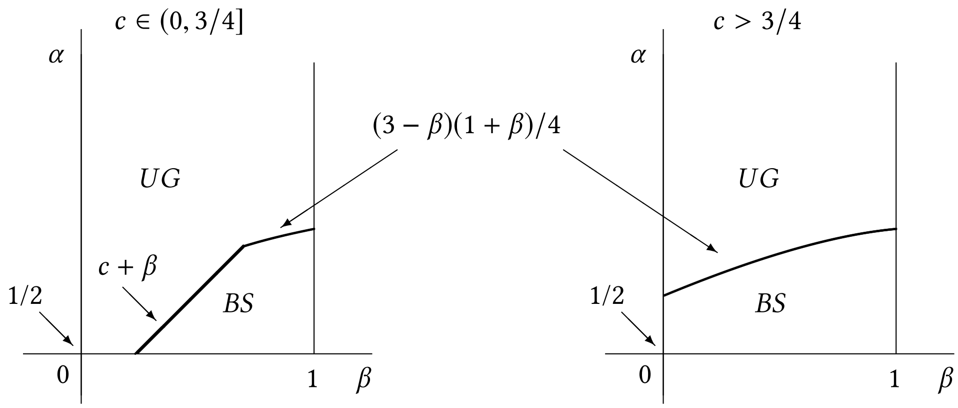

Figure 8: Optimal business model as a function of α and β

5 Antitrust implications

In this section, we show that the entry strategies that arise in equilibrium (Proposition 1) increase welfare relative to the situation prior to entry. Therefore, there is no reason why an antitrust authority should worry about the potential anti-competitive effects of these entry strategies.

Before entry takes place, total welfare in both markets is $w = u + \alpha - c + 3/8$. If the entrant pursues strategy UG (*i.e.*, $\alpha > \min\{(3-\beta)(1+\beta)/4, c+\beta\}$), then total welfare is

$$\begin{aligned} w^{UG} &= u + \alpha - c + \int_{\frac{1+\beta}{2}-\beta}^{1} \left[v - \left(\frac{1+\beta}{2} - \beta \right) \right] dv + \frac{(1+\beta)^2}{4} \\ &= u + \alpha - c + 3\frac{(1+\beta)^2}{8}. \end{aligned}$$

Hence, entry using unbundled pricing with general complementarity enhances welfare because more consumers consume good a, and because consumers purchasing both goods benefit from the complementarity benefit.

If the entrant pursues strategy BS instead (*i.e.*, $\alpha \le \min\{(3-\beta)(1+\beta)/4, c+\beta\}$), then total welfare is $w^{BS} = \int_0^1 (v - c + u + \alpha + \beta)dv = u + \alpha - c + 1/2 + \beta$, and also in this case welfare increases relative to the pre-entry level. Pursuit of entry in market B forces firm 1 to charge a low price for the bundle, such that it cannot exclude any consumer in market A. Firm 1 effectively kills its power to control pricing in market A as a stand-alone monopolist would, which enhances social welfare. Even if firm 2 ends up exiting the market, welfare would increase to $u+\alpha-c+3(1+\beta)^2/8 > u+\alpha-c+3/8$ because of the complementarity gains. Hence, entry using bundling with a specific complementarity is socially beneficial, regardless of whether firm 2 exits the market or not. Proposition 2 follows.

Proposition 2 (Welfare analysis of the optimal business models)**.** *Social welfare increases after entry regardless of the optimal business model used by firm 1.*

Bundling is typically seen as a strategy adopted by incumbents to protect their market from entry by rivals. Regulatory interventions against bundling practices are therefore often advocated, as in the case of Microsoft and Netscape mentioned in the introduction. In contrast, our paper shows that bundling can be used to enter a market where an incumbent holds a strong dominant position sheltered by intense network effects, which is a socially efficient entry strategy if there are complementarities between the bundled products. This happens for online social networks (Facebook, Twitter, Google Plus), cloud storage and software (Dropbox, Google Drive, Apple's iCloud), and customer relationship management software (Salesforce). For example, several iPhone and iPad apps and services rely on data storage on Apple's iCloud (for example, iPhone and iPad backups are stored on iCloud, and a user can also upload each photo she takes automatically to iCloud). These products are in turn highly complementary with Apple software such as Keynote, Numbers and Pages (for example, a user may create a document in Keynote in a mac computer, save it on the iCloud, share it with other users, and access and modify it on her iPhone and iPad).

6 Conclusion

We have studied how an incumbent resting on strong network effects may be replaced by a firm already present in the market for a complementary good. We have considered a variety of entry strategies that can be employed by the entrant to displace the incumbent (and perhaps enhance efficiency in so doing). These entry strategies include bundling and sharing the complementarity benefits with the incumbent.

Our paper may hopefully serve as a building block for future analyses that may enhance our understanding of lateral entry into network goods markets. There are cases to which it is relatively simple to extend our insights. For example, the entrant and the incumbent may produce the network good at different costs, or the entrant need not be a monopolist in the complementary good market. In this cases, it is straightforward to show that the mechanisms studied in this paper will also be at work. Extending our results to other settings, however, is not as straightforward. First, one may study what happens if the incumbent has the possibility to respond by becoming active in the market for complementary goods. Previous works (*e.g.*, Bakos and Brynjolfsson, 2000, or Nalebuff, 2004, *inter alii*) noted that bundle-to-bundle competition can be very intense, so it would be interesting to examine whether this would still hold if bundles include a network good. Second, one may analyze a setting when consumers are not forced to single-home, but rather are allowed to multi-home in the network goods market (as in Choi, 2010). In such cases, the entrant may not compete so fiercely for the incumbent's installed base, so entry might be more difficult. We believe that these issues present interesting directions for future research.

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Appendix

Proof of Lemma 1. We start by constructing the demand correspondence of firm 2. Given that we can restrict attention to $p_a \in [0, 1+\beta]$, we distinguish the following cases that may arise based on the firms' pricing, analyzing the implications of each of them in constructing the firms' demand functions/correspondences:

(a) $U_{b_1} > U_{b_2}$.

(b) $U_{b_1} = U_{b_2}$.

(c) $U_{b_2} > U_{b_1}$.

Regarding case (a), $U_{b_1} > U_{b_2}$ implies that $U_{ab_1}(v) > U_{ab_2}(v)$ for all v (as $\beta \geq 0$), which in turn implies that $n_2 = 0$ must hold. Therefore, $U_{b_1} > U_{b_2}$ if and only if

$$u + \alpha - p_1 > u - p_2.$$

This shows that $n_2 = 0$ can arise as firm 2's quantity demanded whenever $p_2 > p_1 - \alpha$ is met.

As for case (b), $U_{b_1} = U_{b_2}$ implies that $U_{ab_1}(v) > U_{ab_2}(v)$ for all v whenever $\beta > 0$ (we treat $\beta = 0$ separately further below). Those consumers with valuation greater than $p_a - \beta$ buy both a and b_1, whereas those consumers with valuation smaller than $p_a - \beta$ do not buy a and are indifferent between b_1 and b_2. If $p_a \leq \beta$, then $n_2 = 0$ must hold. If instead $p_a > \beta$, the measure of consumers who buy only b_2 cannot exceed the total number of consumers who do not buy a and b_1, which is equal to $p_a - \beta$. Hence, $0 \leq n_2 \leq p_a - \beta$ must hold, with $U_{b_1} = U_{b_2}$ yielding that it must also hold that

$$p_2 = p_1 + (2n_2 - 1)\alpha.$$

Note that $n_2 = 0$ is also consistent with $p_a > \beta$. Then, we have shown that $n_2 = 0$ can arise as firm 2's quantity demanded if $p_2 = p_1 - \alpha$, and

$$n_2 = \frac{p_2 - p_1 + \alpha}{2\alpha} \in (0, p_a - \beta]$$

can arise as firm 2's quantity demanded if $p_a > \beta$ and $p_1 - \alpha < p_2 \leq p_1 - \alpha + 2\alpha(p_a - \beta)$.

When $\beta = 0$ in case (b), note that $U_{b_1} = U_{b_2}$ implies that $U_{ab_1}(v) = U_{ab_2}(v)$ for all v. Those consumers with valuation greater than p_a purchase a and are indifferent between buying b_1 and b_2 as well, whereas those consumers with valuation smaller than p_a do not purchase a and are indifferent between b_1 and b_2. Clearly, as $0 \leq n_2 \leq 1$, when $U_{b_1} = U_{b_2}$ it must also hold that $p_2 = p_1 + (2n_2 - 1)\alpha$, so

$$n_2 = \frac{p_2 - p_1 + \alpha}{2\alpha} \in (0, 1]$$

can arise as firm 2's quantity demanded if $p_1 - \alpha \leq p_2 \leq p_1 + \alpha$ whenever $\beta = 0$.

Finally, with regards to case (c) $U_{b_2} > U_{b_1}$ implies that $p_2 < p_1 + (2n_2 - 1)\alpha$ must be satisfied. Depending on the comparison between $U_{ab_2}(v)$ and $U_{ab_1}(v)$, we have three subcases:

(c1) $U_{ab_2}(v) > U_{ab_1}(v)$ for all v.

(c2) $U_{ab_2}(v) = U_{ab_1}(v)$ for all v.

(c3) $U_{ab_2}(v) < U_{ab_1}(v)$ for all v.

In subcase (c1), $U_{b_2} > U_{b_1}$ and $U_{ab_2}(v) > U_{ab_1}(v)$ for all v; no consumer wants to buy b_1, and therefore $n_2 = 1$. Inequality $U_{ab_2}(v) > U_{ab_1}(v)$ implies $p_2 < p_1 + (2n_2 - 1)\alpha - \beta$. Note that this condition is consistent with the one for $U_{b_2} > U_{b_1}$, which is $p_2 < p_1 + (2n_2 - 1)\alpha$. Given that $n_2 = 1$ must hold, this subcase arises if $p_2 < p_1 + \alpha - \beta$.

In subcase (c2), $U_{ab_2}(v) = U_{ab_1}(v)$ implies $p_2 = p_1 + (2n_2 - 1)\alpha - \beta$. Recalling that $U_{b_1} < U_{b_2}$, $U_{b_2} > U_{ab_2}(v) = U_{ab_1}(v)$ would imply that any consumer with valuation smaller than $\min\{1, p_a\}$ should purchase b_2, whereas any other consumer should purchase a together with either b_1 or b_2. Therefore, the measure of consumers who buy only b_2 must be greater than or equal to $\min\{1, p_a\}$, and any

$$n_2 = \frac{p_2 - p_1 + \alpha + \beta}{2\alpha} \in [\min\{1, p_a\}, 1]$$

can arise as firm 2's quantity demanded if $p_2 = p_1 + (2n_2 - 1)\alpha - \beta$ holds. Note that $n_2 = 1$ is consistent with fulfilled expectations if $p_2 = p_1 + \alpha - \beta$, and $n_2 \in [p_a, 1)$ is consistent if $p_1 + 2\alpha p_a - \alpha - \beta \leq p_2 < p_1 + \alpha - \beta$, and $0 \leq p_a < 1$.

In (c3), $U_{ab_2}(v) < U_{ab_1}(v)$ implies that $p_1 + (2n_2 - 1)\alpha - \beta < p_2 < p_1 + (2n_2 - 1)\alpha$. Let

$$\widehat{v} \equiv p_a - \beta + (2n_2 - 1)\alpha + p_1 - p_2$$

denote the unique value of v such that $U_{b_2} = U_{ab_1}(v)$ holds. Note that $\widehat{v} \in (p_a - \beta, p_a)$, and as $p_a - \beta$ may be negative, we need to consider both the case in which $\widehat{v}$ is below 0 and the case in which it is above 0. Similarly, as p_a may exceed 1, we need to consider both the case in which $\widehat{v}$ is above 1 and the case in which it is below 1.

Firstly, suppose that $\widehat{v} \leq 0$ (which necessarily requires $p_a < \beta$), so that $p_a - \beta + (2n_2 - 1)\alpha + p_1 - p_2 \leq 0$. Satisfaction of rational expectations by consumers (as required by Nash equilibrium behavior) implies that we must have $n_2 = 0$, so the condition becomes $p_1 + p_a - \alpha - \beta \leq p_2$, which must hold at the same time as $p_1 - \alpha - \beta < p_2 < p_1 - \alpha$, so we simply need that $p_1 + p_a - \alpha - \beta \leq p_2 < p_1 - \alpha$.

Secondly, suppose that $\widehat{v} \geq 1$ (which necessarily requires that $p_a \geq 1$, as $U_{b_2} > U_{ab_2}(v)$ for all $v \in [0, 1]$), so that $p_2 \leq p_1 + (2n_2 - 1)\alpha + p_a - \beta - 1$. In order for consumers expectations to be fulfilled, it must hold that $n_2 = 1$, so the condition becomes $p_2 \leq p_1 + \alpha + p_a - \beta - 1$. Recall we also need $p_1 + (2n_2 - 1)\alpha - \beta < p_2 < p_1 + (2n_2 - 1)\alpha$, but given that we only consider cases such that $p_a \leq 1 + \beta$, the final restriction is $p_1 + \alpha - \beta < p_2 \leq p_1 + \alpha - (1 + \beta - p_a)$.

Lastly, suppose now that $\widehat{v} \in (0, 1)$: $p_1 + p_a + (2n_2 - 1)\alpha - \beta - 1 < p_2 < p_1 + p_a + (2n_2 - 1)\alpha - \beta$. Then n_2 must equal the measure of consumers whose valuation is smaller than $\widehat{v}$ (as $U_{b_2} > U_{ab_1}(v) > U_{ab_2}(v)$ and $U_{b_2} > U_{b_1}$ for such consumers). That is,

$n_2 = \Pr(v \leq \widehat{v})$ must hold in order for consumers expectations to be fulfilled (as required by Nash equilibrium behavior), so the fact that $\Pr(v \leq \widehat{v}) = \widehat{v}$ implies that we must have $n_2 = (2n_2 - 1)\alpha + p_a - \beta + p_1 - p_2$, which yields

$$n_2 = \frac{p_2 - p_1 - p_a + \alpha + \beta}{2\alpha - 1}.$$

The constraints that $p_1 + p_a - 1 + (2n_2 - 1)\alpha - \beta < p_2 < p_1 + p_a + (2n_2 - 1)\alpha - \beta$ and $p_1 + (2n_2 - 1)\alpha - \beta < p_2 < p_1 + \alpha(2n_2 - 1)$ imply that $\max\{0, p_a - \beta\} < n_2 < \min\{1, p_a\}$. Because we assumed that $\alpha > 1/2$, we must have $p_1 + p_a - \alpha - \beta + (2\alpha - 1)\max\{p_a - \beta, 0\} < p_2 < p_1 + p_a - \alpha - \beta + (2\alpha - 1)\min\{p_a, 1\}$.

To summarize, we have shown that firm 2's demand correspondence is as follows:

- $n_2 = 0$ if $p_2 \geq p_1 - \alpha - \max\{\beta - p_a, 0\}$.
- $n_2 = 1$ if $p_2 \leq p_1 + \alpha - \beta + \max\{p_a - 1, 0\}$.
- $n_2 = \dfrac{p_2 - p_1 + \alpha}{2\alpha}$ if $p_1 - \alpha < p_2 \leq p_1 - \alpha + 2\alpha(p_a - \beta)$ and $p_a \in (\beta, 1 + \beta)$.
- $n_2 = \dfrac{p_2 - p_1 + \alpha + \beta}{2\alpha}$ if $p_1 + 2\alpha p_a - \alpha - \beta \leq p_2 < p_1 + \alpha - \beta$ and $p_a \in [0, 1)$.
- $n_2 = \dfrac{p_2 - p_1 - p_a + \alpha + \beta}{2\alpha - 1}$ if $p_1 + p_a - \alpha - \beta + (2\alpha - 1)\max(p_a - \beta, 0) < p_2 \leq p_1 + p_a - \alpha - \beta + (2\alpha - 1)\min\{p_a, 1\}$.

This demand correspondence is represented in the left panels of Figure 3. The right panels of this figure identify the demand function of firm 2 by using the refinement equilibrium that we adopted (see Section 2). We now proceed to demonstrate the results appearing in Lemma 1.

Denote equilibrium values using the "US" superscript. Let us try to sustain $n_2^{US} = 1$ in equilibrium. Clearly, firm 1 must be charging the monopoly price in market A, so $p_a^{US} = 1/2$, and it must be earning $\pi_1^{US} = 1/4$. Also, firm 2 must be charging the highest possible price such that its demand equals 1. If firm 1 is charging p_1^{US} in market B, then firm 2 must be charging $p_2^{US} = p_1^{US} + \alpha - \beta$ (see top and middle panels on the right of Figure 3). Suppose first that $p_1^{US} > 0$. If firm 1 deviates by charging $\widehat{p}_a < 1$, then the best it can do is to charge $\widehat{p}_1 = p_1^{US} - \epsilon$ for $\epsilon > 0$ small enough so as to attract all consumers in market B and attract those consumers in market A whose valuation exceeds $\widehat{p}_a - \beta$. Doing so yields

$$\begin{aligned} \pi_1(\widehat{p}_1, \widehat{p}_a) &= \widehat{p}_1 - c + \widehat{p}_a[\min\{1, 1 + \beta - \widehat{p}_a\}] \\ &= \begin{cases} \widehat{p}_1 - c + \widehat{p}_a & \text{if } \widehat{p}_a \leq \beta \\ \widehat{p}_1 - c + \widehat{p}_a(1 + \beta - \widehat{p}_a) & \text{if } \widehat{p}_a \in (\beta, 1) \end{cases}. \end{aligned}$$

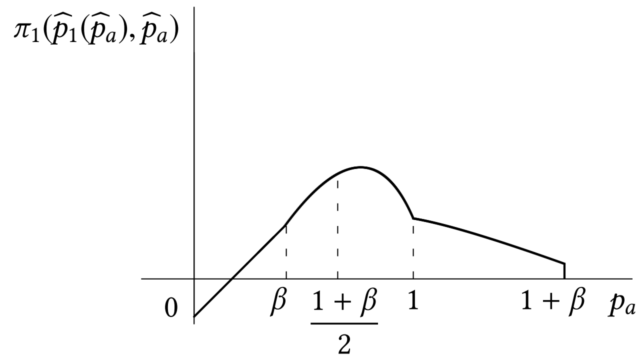
Consider now what happens if firm 1 chooses $\widehat{p}_a \geq 1$. Clearly the best that firm 1 could do is to charge $\widehat{p}_1(\widehat{p}_a) = p_1^{US} - (\widehat{p}_a - 1) - \epsilon$ so as to attract all consumers in market B

(see the bottom right panel of Figure 3) and attract those consumers in market A whose valuation exceeds $\widehat{p}_a - \beta$. This can be accomplished if $\widehat{p}_a < p_1^{US} + 1$, as the price charged for b_1 would not be negative; however, if $\widehat{p}_a \geq p_1^{US} + 1$, then it is impossible for firm 1 to get some demand in market B.[11] Taking into account that the non-negativity of good b_1's price limits firm 1's deviations so that $\widehat{p}_a < p_1^{US} + 1$, firm 1 would earn

$$\pi_1(\widehat{p}_1(\widehat{p}_a), \widehat{p}_a) = p_1^{US} - c + 1 - \widehat{p}_a + \widehat{p}_a[\min\{1, 1 + \beta - \widehat{p}_a\}]$$

when charging $\widehat{p}_a \in [1, p_1^{US} + 1)$. Because $\beta < 1$, firm 1's payoff to deviating as a function of $\widehat{p}_a$ is represented in Figure A.

Figure A



It follows that the optimal deviation for firm 1 involves $\widehat{p}_1^{US} = p_1^{US} - \epsilon > 0$ and $\widehat{p}_a^{US} = (1+\beta)/2$ so as to earn $\pi_1(\widehat{p}_1^{US}, \widehat{p}_a^{US}) = p_1^{US} - c - \epsilon + (1+\beta)^2/4$. Recalling that we have made the working assumption that $p_1^{US} > 0$, firm 1 does not deviate if $1/4 \geq p_1^{US} - c + (1+\beta)^2/4$, that is, $p_1^{US} \leq c - \beta(2+\beta)/4$, with $c > \beta(2+\beta)/4$ in order for $p_1^{US} > 0$. In an equilibrium not sustained by weakly dominated strategies,[12] we must have

$$p_1^{US} = c - \beta(2+\beta)/4 > 0;$$

in such a case,

$$p_2^{US} = \alpha - \beta + c - \beta(2+\beta)/4 = c + \alpha - \beta(6+\beta)/4,$$

and therefore we need $\alpha \geq \beta(6+\beta)/4$ so that firm 2 has no incentive to perform a

[11] If $\widehat{p}_a \geq p_1^{US} + 1$, we should have $\widehat{p}_1 = 0$ and $\widehat{p}_a = 1 + \beta - \alpha + p_2^{US} - \epsilon = 1 + p_1^{US} - \epsilon$ in order for firm 1 to attract market B's demand, but $\widehat{p}_a = 1 + p_1^{US} - \epsilon < 1 + p_1^{US}$ contradicts $\widehat{p}_a \geq p_1^{US} + 1$.

[12] It is standard not to allow for weakly dominated strategies in a Bertrand pricing game with asymmetric firms. To give a simple example, consider a Bertrand duopoly where firms bear a marginal cost of production equal to 1 and compete in prices to attract a single unit-demand consumer who values the product of one of the firms at 5 and the product of the other firm at 4. When the smallest monetary unit is $\epsilon > 0$ (where ϵ is not zero but is "small"), that the former firm prices at $1.5 - \epsilon$ and the latter prices at $0.5 < 1$ is certainly a Nash equilibrium, but it is sustained by a weakly dominated strategy. The only Nash equilibrium not sustained by weakly dominated strategies converges as $\epsilon \to 0$ to the well-known equilibrium in which the vertically superior firm charges 2 and the rival sells at marginal cost. Our focus will always be on the kind of equilibrium equivalent to this type in our setting. See Griva and Vettas (2011) for a similar restriction in a setting with direct network effects.

unilateral deviation. We conclude the analysis of $n_2 = 1$ by considering the case in which $p_1^{US} = 0$. Clearly, we must have $p_2^{US} = \alpha - \beta$ in such situations. Firm 1 cannot undertake any profitable deviation, and ruling out deviations by firm 2 simply requires that $c \leq \alpha - \beta$, or $\alpha \geq \beta + c$.

Let us now try to sustain $n_2 = 0$ in equilibrium with $p_a^{US} < 1$. In such a case, it must hold that $p_1^{US} = p_2^{US} + \beta - \alpha$. Also, firm 2 should have no incentive to charge $p_2^{US} - \epsilon$, so it must hold that $p_2^{US} \leq c$, which implies that $p_2^{US} = c$ in an equilibrium not sustained by weakly dominated strategies. Therefore,

$$p_1^{US} = c + \beta - \alpha,$$

whereas the price that firm 1 charges in market A must be

$$p_a^{US} = \arg\max_{p_a} \left\{ p_1^{US} - c + p_a[\min\{1, 1 + \beta - p_a\}] \right\}.$$

Because $\beta < 1$, then $p_a^{US} = (1 + \beta)/2$, so $p_a^{US} < 1$ does hold, as initially claimed. Firm 1 then earns the following equilibrium profit:

$$\pi_1^{US} = \beta - \alpha + (1 + \beta)^2 / 4.$$

Besides $p_1^{US} \geq 0$ (*i.e.*, $c \geq \alpha - \beta$), firm 1 should have no incentive to focus on monopolizing market A only, in which case it would earn $1/4$. It follows that $\beta - \alpha + (1 + \beta)^2 / 4 \geq 1/4$ requires that $\alpha \leq \beta (6 + \beta) / 4$. Firm 1 should have no incentive either to charge $\widehat{p}_a \geq 1$ and set $\widehat{p}_1(\widehat{p}_a)$ in such a way that $\widehat{p}_1(\widehat{p}_a) = p_2^{US} + \beta - \alpha - (\widehat{p}_a - 1)$, provided $\widehat{p}_1(\widehat{p}_a) > 0$, because otherwise the deviation cannot be profitable. Doing so yields

$$\begin{aligned} \pi_1(\widehat{p}_1(\widehat{p}_a), \widehat{p}_a) &= \widehat{p}_1(\widehat{p}_a) - c + \widehat{p}_a[\min\{1, 1 + \beta - \widehat{p}_a\}] \\ &= p_2^{US} - c + \beta - \alpha + 1 - \widehat{p}_a + \widehat{p}_a[\min\{1, 1 + \beta - \widehat{p}_a\}] \\ &= p_2^{US} - c + 1 + \beta - \alpha + \widehat{p}_a(\beta - \widehat{p}_a), \end{aligned}$$

because $\widehat{p}_a \geq 1 > \beta$. The payoff to deviating is maximized at $\widehat{p}_a^{US} = 1$ (note that $\pi_1(\widehat{p}_1(\widehat{p}_a), \widehat{p}_a)$ decreases with $\widehat{p}_a$ whenever $\widehat{p}_a \geq 1$), so we need that

$$\pi_1^{US} = \beta - \alpha + (1 + \beta)^2 / 4 \geq 1 + \beta - \alpha + \beta - 1,$$

which always holds.

Finally, let us try to sustain an equilibrium in which $n_2 = 0$ and $p_a^{US} \geq 1$. In such a case, it must hold that firm 1 sells good b_1 at price $p_1^{US}(p_a^{US}) = p_2^{US} + \beta - \alpha - (p_a^{US} - 1) \geq 0$. Obviously, $p_2^{US} = c$ in an equilibrium not sustained by weakly dominated strategies, and

$$p_a^{US} \in \arg\max_{\widehat{p}_a \leq c + 1 + \beta - \alpha} \left\{ p_1^{US}(\widehat{p}_a) - c + \widehat{p}_a[\min\{1, 1 + \beta - \widehat{p}_a\}] \right\}.$$

Because $\beta < 1$, we should have $p_a^{US} = \min\{\beta/2, c + 1 + \beta - \alpha\} < 1$, which contradicts $p_a^{US} \geq 1$. It follows that there cannot exist an equilibrium with $n_2 = 0$ and $p_a^{US} \geq 1$.

■

Proof of Lemma 2. We start by constructing firm 2's demand correspondence. To do so, we consider the following cases: (a) $U_{b_1} > U_{b_2}$, (b) $U_{b_1} = U_{b_2}$, and (c) $U_{b_2} > U_{b_1}$.

Regarding case (a), $U_{b_1} > U_{b_2}$ implies that $U_{ab_1}(v) > U_{ab_2}(v)$ for all v, which in turn implies that $n_2 = 0$. Therefore, we have that $U_{b_1} > U_{b_2}$ if and only if

$$u + \alpha - p_1 > u - p_2.$$

This shows that $n_2 = 0$ can arise as firm 2's quantity demanded whenever $p_2 > p_1 - \alpha$ is met.

As for case (b), $U_{b_1} = U_{b_2}$ implies that $U_{ab_1}(v) = U_{ab_2}(v)$ for all v. Those consumers with valuation greater than $p_a - \beta$ purchase a together with either b_1 or b_2, whereas those consumers with valuation smaller than $p_a - \beta$ do not purchase a and are indifferent between b_1 and b_2. as $U_{b_1} = U_{b_2}$ yields that it must hold that

$$p_2 = p_1 + (2n - 1)\alpha,$$

with $0 \leq n_2 \leq 1$, we have then shown that $n_2 = 0$ can arise as firm 2's quantity demanded if $p_2 = p_1 - \alpha$, $n_2 = 1$ can arise as firm 2's quantity demanded if $p_2 = p_1 + \alpha$, and

$$n_2 = \frac{p_2 - p_1 + \alpha}{2\alpha}$$

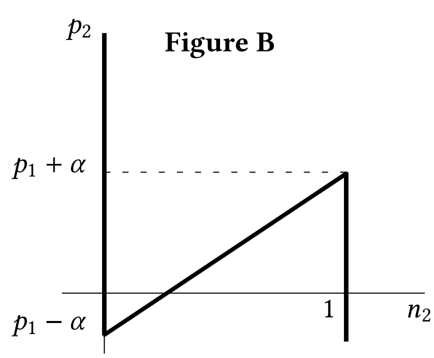
if $p_1 - \alpha < p_2 < p_1 + \alpha$.

Finally, with regards to case (c), $U_{b_2} > U_{b_1}$ implies that $U_{ab_2}(v) > U_{ab_1}(v)$ for all v, so no consumer wants to buy b_1, and therefore $n_2 = 1$. Also, $U_{b_2} > U_{b_1}$ yields that $p_2 < p_1 + (2n_2 - 1)\alpha$ must be satisfied. Given that $n_2 = 1$, this case holds if $p_2 < p_1 + \alpha$.

To sum up, we have shown that firm 2's demand correspondence is as follows:

- $n_2 = 0$ for $p_2 \geq p_1 - \alpha$.
- $n_2 = 1$ for $p_2 \leq p_1 + \alpha$.
- $n_2 = \dfrac{p_2 - p_1 + \alpha}{2\alpha}$ for $p_1 - \alpha < p_2 < p_1 + \alpha$.

This demand correspondence is represented in Figure B.



What remains to be proved appears in the discussion that precedes the statement of this lemma.

■

Proof of Lemma 3. In order to find out firm 2's demand correspondence, note that charging $p < 0$ is weakly dominated, so we focus on $p \geq 0$ in what follows. We cover all the possible cases that can arise:

- Suppose $U_{ab_1}(v) > U_{ab_2}(v)$ and $U_{ab_1}(v) > U_{b_2}$ for all $v \in [0, 1]$. Then $n_2 = 0$ whenever $p_2 > -\alpha - \beta$ and $p_2 > -(v - p) - \alpha - \beta$ for $v = 0$, that is, whenever $p_2 > p - \alpha - \beta$. Because $p \geq 0$, it holds that $n_2 = 0$ whenever $p_2 - p > -\alpha - \beta$.

- Suppose $U_{b_2} \geq U_{ab_1}(v)$ and $U_{b_2} > U_{ab_2}(v)$ for all $v \in [0, 1]$. Then $n_2 = 1$ and $p > v$ for all $v \in [0, 1]$ as well as $\alpha - \beta - p_2 \geq v - p$ for all $v \in [0, 1]$. Hence, $n_2 = 1$ can arise whenever $p \geq 1$ and $p_2 - p \leq \alpha - \beta - 1$. Firm 1 makes no sale of good a.

- Suppose $U_{ab_2}(v) > U_{ab_1}(v)$ for all $v \in [0, 1]$. Then it must hold that $n_2 = 1$ and $p_2 < \alpha - \beta$. Those consumers with $v > p$ purchase both a and b_2, and those with $v \leq p$ purchase b_2 only. For $p \leq 1$, firm 1 makes sales of good a equal to $1 - p \in [0, 1]$.

- Suppose $U_{ab_1}(v) = U_{ab_2}(v)$ for all $v \in [0, 1]$. Then we must have $\alpha(1 - n) + \beta = \alpha n_2 - p_2$. If there exists $\widehat{v} \in (0, 1)$ such that $U_{ab_2}(v) > U_{b_2}$ for $v > \widehat{v}$ and $U_{ab_2}(v) < U_{b_2}$ for $v < \widehat{v}$, then it must be the case that $p \in (0, 1)$ so that $\widehat{v} = p$. Then

$$n_2 = \frac{p_2 + \alpha + \beta}{2\alpha} \in [0, 1]$$

  for $-\alpha - \beta \leq p_2 \leq \alpha - \beta$. We should also have that $p_2 \geq 2\alpha p - \alpha - \beta$ in order for $\widehat{v} \leq n_2$ (so that consumer expectations are fulfilled), so $p \in (0, 1)$ implies that it should hold that $2\alpha p - \alpha - \beta \leq p_2 \leq \alpha - \beta$. Firm 1 makes sales of good a equal to $1 - p$. If instead $\widehat{v} = 1$, which means that $p \geq 1$, then firm 1 makes no sale of good a and on top of that $n_2 = 1$. If instead $\widehat{v} = 0$, which means that $p = 0$, then firm 1 makes sales of good a equal to 1.

- Suppose that $U_{ab_1}(v) > U_{ab_2}(v)$ for all $v \in [0, 1]$ and that there exists $\widehat{v} \in (0, 1)$ such that $U_{ab_1}(v) > U_{b_2}$ for $v > \widehat{v}$ and $U_{ab_1}(v) < U_{b_2}$ for $v < \widehat{v}$. In this case $\widehat{v} - p + \alpha(1 - n) + \beta = \alpha n_2 - p_2$, with $\widehat{v} = n_2$ because of rational expectations, so $\alpha - p + p_2 + \beta = n_2(2\alpha - 1)$. Therefore, $\alpha > 1/2$ implies that

$$n_2 = \frac{p_2 - p + \alpha + \beta}{2\alpha - 1} \in (0, 1)$$

  for $p - \alpha - \beta < p_2 < p + \alpha - \beta - 1$. We also need $\alpha n_2 - p_2 < \alpha(1 - n) + \beta$, that is, $p_2 < 2\alpha p - \alpha - \beta$. Therefore, the condition that must be satisfied is $p - \alpha - \beta < p_2 < \min\{2\alpha p - \alpha - \beta, p + \alpha - \beta - 1\}$. Firm 1 makes sales of a equal to $1 - n$.

To sum up, we have proven that firm 2's demand correspondence is as follows:

- $n_2 = 0$ for $p_2 > p - \alpha - \beta$. In this case, firm 2 makes no sales of good b_2 and firm 1's sales of the bundled good equal 1.
- $n_2 = 1$ for $p < 1$ and $p_2 \leq \alpha - \beta$. In this case, firm 2's sales of good b_2 equal 1 and firm 1's sales of the bundled good equal $1 - p$.
- $n_2 = 1$ for $p \geq 1$ and $p_2 \leq \alpha - \beta + p - 1$. In this case, firm 2's sales of good b_2 equal 1 and firm 1's sales of the bundled good equal 0.
- $n_2 = \dfrac{p_2 + \alpha + \beta}{2\alpha} \in [0, 1]$ for $2\alpha p - \alpha - \beta \leq p_2 \leq \alpha - \beta$, which requires that $p \in (0, 1)$. In this case, firm 2 makes n_2 sales of good b_2, and firm 1's sales of the bundled good equal $1 - p$.
- $n_2 = \dfrac{p_2 - p + \alpha + \beta}{2\alpha - 1} \in [0, 1]$ for $p - \alpha - \beta < p_2 < 2\alpha p - \alpha - \beta$ if $p \in (0, 1)$ and for $p - \alpha - \beta < p_2 < p + \alpha - \beta - 1$ if $p \geq 1$. In this case, firm 2 makes n_2 sales of good b_2, and firm 1's sales of the bundled good equal $1 - n_2$.

Firm 2's demand correspondence is represented in Figure C, in which we distinguish two cases, depending on the value of p.

Figure C

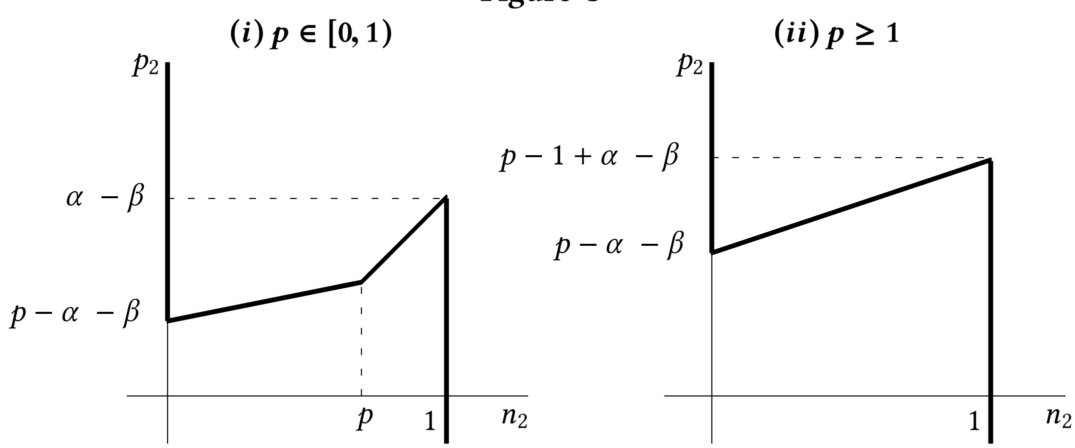


Figure 6 in Subsection 3.3 represents the demand function that results from adopting the equilibrium refinement adopt in this paper. What remains to be proved appears in the discussion that precedes the statement of this lemma.

■

Proof of Lemma 4. In order to find out firm 2's demand correspondence, note that charging $p < 0$ is weakly dominated, so we focus on $p \geq 0$ in what follows. We cover all the possible cases that can arise:

- Suppose $U_{ab_1}(v) > U_{ab_2}(v)$ and $U_{ab_1}(v) > U_{b_2}$ for all $v \in [0,1]$. Then $n_2 = 0$ whenever $p_2 > -\alpha$ and $p_2 > -(v-p) - \alpha - \beta$ for $v = 0$, that is, whenever $p_2 > p - \alpha - \beta$. So it holds that $n_2 = 0$ whenever $p \geq \beta$ and $p_2 - p > -\alpha - \beta$ or whenever $p < \beta$ and $p_2 > -\alpha$.

- Suppose $U_{b_2} \geq U_{ab_1}(v)$ and $U_{b_2} > U_{ab_2}(v)$ for all $v \in [0,1]$. Then $n_2 = 1$ and $p > v + \beta$ for all $v \in [0,1]$ as well as $\alpha - \beta - p_2 \geq v - p$ for all $v \in [0,1]$. Hence, $n_2 = 1$ can arise whenever $p \geq 1 + \beta$ and $p_2 - p \leq \alpha - \beta - 1$. Firm 1 makes no sale of good a.

- Suppose $U_{ab_2}(v) > U_{ab_1}(v)$ for all $v \in [0,1]$. Then it must hold that $n_2 = 1$ and $p_2 < \alpha$. Those consumers with $v > p - \beta$ purchase both a and b_2, and those with $v \leq p - \beta$ purchase b_2 only. For $\beta < p \leq 1 + \beta$, firm 1 makes sales of good a equal to $1 + \beta - p \in [0,1]$, whereas for $p \leq \beta$, firm 1 makes sales of good a equal to 1.

- Suppose $U_{ab_1}(v) = U_{ab_2}(v)$ for all $v \in [0,1]$. Then we must have $\alpha(1-n_2) = \alpha n_2 - p_2$. If there exists $\widehat{v} \in (0,1)$ such that $U_{ab_2}(v) > U_{b_2}$ for $v > \widehat{v}$ and $U_{ab_2}(v) < U_{b_2}$ for $v < \widehat{v}$, then it must be the case that $p \in (\beta, 1+\beta)$ so that $\widehat{v} = p - \beta$. Then

$$n_2 = \frac{p_2 + \alpha}{2\alpha} \in [0,1]$$

  for $-\alpha \leq p_2 \leq \alpha$. We should also have that $p_2 \geq 2\alpha p - \alpha - 2\alpha\beta$ in order for $\widehat{v} \leq n_2$ (so that consumer expectations are fulfilled), so $p \in (\beta, 1+\beta)$ implies that it should hold that $2\alpha p - \alpha - 2\alpha\beta \leq p_2 \leq \alpha$. Firm 1 makes sales of good a equal to $1 + \beta - p$. If instead $\widehat{v} = 1$, which means that $p \geq 1 + \beta$, then firm 1 makes no sale of good a and on top of that $n_2 = 1$. If instead $\widehat{v} = 0$, which means that $p \leq \beta$, then firm 1 makes sales of good a equal to 1.

- Suppose that $U_{ab_1}(v) > U_{ab_2}(v)$ for all $v \in [0,1]$ and that there exists $\widehat{v} \in (0,1)$ such that $U_{ab_1}(v) > U_{b_2}$ for $v > \widehat{v}$ and $U_{ab_1}(v) < U_{b_2}$ for $v < \widehat{v}$. In this case $\widehat{v} - p + \alpha(1-n) + \beta = \alpha n_2 - p_2$, with $\widehat{v} = n_2$ because of rational expectations, so $\alpha - p + p_2 + \beta = n_2(2\alpha - 1)$. Therefore, $\alpha > 1/2$ implies that

$$n_2 = \frac{p_2 - p + \alpha + \beta}{2\alpha - 1} \in (0,1)$$

  for $p - \alpha - \beta < p_2 < p + \alpha - \beta - 1$. We also need $\alpha n_2 - p_2 < \alpha(1-n_2)$, that is, $p_2 < 2\alpha p - \alpha$. Therefore, the condition that must be satisfied is $p - \alpha - \beta < p_2 < \min\{2\alpha p - \alpha, p + \alpha - \beta - 1\}$. Firm 1 makes sales of a equal to $1 - n_2$.

To sum up, we have proven that firm 2's demand correspondence is as follows:

- $n_2 = 0$ for $p_2 > -\alpha + \max\{p - \beta, 0\}$. In this case, firm 2 makes no sales of good b_2 and firm 1's sales of the bundled good equal 1.

- $n_2 = 1$ for $p \geq 1 + \beta$ and $p_2 \leq \alpha - \beta + p - 1$. In this case, firm 2's sales of good b_2 equal 1 and firm 1's sales of the bundled good equal 0.

- $n_2 = 1$ for $p < 1 + \beta$ and $p_2 \leq \alpha$. In this case, firm 2's sales of good b_2 equal 1 and firm 1's sales of the bundled good equal $1 + \beta - p$ if $p > \beta$, and equal 1 if $p \leq \beta$.

- $n_2 = \dfrac{p_2 + \alpha}{2\alpha} \in [0, 1]$ for $2\alpha p - \alpha - 2\alpha\beta \leq p_2 \leq \alpha$, which requires that $p \in (\beta, 1 + \beta)$. In this case, firm 2 makes n sales of good b_2, and firm 1's sales of the bundled good equal $1 + \beta - p$.

- $n_2 = \dfrac{p_2 - p + \alpha + \beta}{2\alpha - 1} \in [0, 1]$ for $p - \alpha - \beta < p_2 < \min\{2\alpha p - \alpha, p + \alpha - \beta - 1\}$. Firm 2 makes n sales of good b_2, and firm 1's sales of the bundled good equal $1 - n_2$.

What remains to be proved appears in the discussion that precedes the statement of this lemma.

■

Proof of Proposition 1. Based on Lemmas 1 and 3, it is straightforward to show that $\pi_1^{BS} \geq \pi_1^{US}$ when $\alpha \leq \min\{3/4, c\} + \beta$, whereas $\pi_1^{UG} \geq \pi_1^{US}$ when $\alpha \geq \beta$ (from Lemmas 1 and 2). It is therefore immediate to conclude that unbundled pricing with a specific complementarity can never be optimal, given that $\beta < \min\{3/4, c\} + \beta$. We use a similar argument to eliminate bundling with a general complementarity, even though it requires more calculations. Indeed, it holds using Lemmas 2 and 4 that $\pi_1^{UG} > \pi_1^{BG}$ when $\alpha > \min\{c, (3 - \beta)(1 + \beta)/4\}$, so we focus on comparing π_1^{BG} with π_1^{BS}. As $\pi_1^{BS} = \pi_1^{BG}$ when $\alpha \leq \min\{c, 1 + \beta\}$, we break the indifference by assuming that there is some (arbitrarily small) cost of modifying firm 1's products to make the complementarity general, so we take π_1^{BS} to be greater than π_1^{BG} when $\alpha \leq \min\{c, 1 + \beta\}$. If $\alpha > \min\{c, 1 + \beta\}$, $\pi_1^{BS} > \pi_1^{BG}$ when $c \geq 1 + \beta$, and therefore pure bundling with a general complementarity can never be optimal. When $c < 1 + \beta$, $\pi_1^{BS} > \pi_1^{BG}$ if and only if $\alpha < \min\{1 + \beta - (1 - c + \beta)^2/4, \min\{1, c\} + \beta\}$. Yet, pure bundling with a general complementarity can be discarded also in this case because: (i) $\min\{c, (3 - \beta)(1 + \beta)/4\} < \min\{1 + \beta - (1 - c + \beta)^2/4, \min\{1, c\} + \beta\}$ when $c < 1 + \beta$; and (ii) $\pi_1^{UG} > \pi_1^{BG}$ in the parameter region in which $\pi_1^{BG} > \pi_1^{BS}$, namely, $\alpha > \min\{1 + \beta - (1 - c + \beta)^2/4, \min\{1, c\} + \beta\}$. As a consequence, pure bundling with a general complementarity cannot represent the optimal decision for firm 1, which is therefore left with the decision between unbundled pricing with a general complementarity and pure bundling with a specific complementarity. Proposition 1 directly follows from comparing π_1^{UG} with π_1^{BS}.

■

Online Appendix: Mixed bundling (not for publication)

The purpose of this section is to show that any equilibrium arising under mixed bundling is characterized by the same profit or less for firm 1 than the unique equilibrium of a game in which it chooses between unbundled pricing and pure bundling. We confine our demonstration to the case of a specific complementarity but similar results emerge for the case of a general complementarity. Our results parallel Whinston's (1990) findings without network effects.

With mixed bundling, firm 1 offers consumers three choices: they can buy product a alone and pay a price p_a, they can buy product b_1 and pay a price p_1, or they can buy the bundle formed by products a and b_1 at a discount price of $p_a + p_1 - d$, where d is an endogenously chosen discount such that $d \geq 0$. Because of free disposal, consumers may buy a, b_1 and b_2 with the aim of using a and b_2, in order to obtain an effective discount equal to $d - p_1$ (provided $d > p_1$, of course). In that case, however, consumers forgo the complementarity benefit $\beta \geq 0$ and the network benefits provided by good b_1.

The utility derived by consuming good b_i is still given by (1). With mixed bundling, the utility generated by consuming a and b_1 is instead given by:

$$U_{ab_1}(v) = v - p_a + \beta + d + u + \alpha n_1^e - p_1, \tag{9}$$

whereas the utility generated by consuming a and b_2 is

$$U_{ab_2}(v) = v - p_a + u + \alpha n_2^e - p_2. \tag{10}$$

Finally, the utility generated by purchasing all goods and disposing of good b_1 is

$$U_{ab_1b_2}(v) = v - p_a + d + u + \alpha n_2^e - p_1 - p_2. \tag{11}$$

Recall we assume u is large enough, so that market B is fully covered To find out firm 2's demand correspondence, it is worth noting first that, regardless of v, $U_{ab_2}(v) \geq U_{ab_1b_2}(v)$ for all v if and only if $d \leq p_1$, so we distinguish two situations depending on whether $p_1 - d$ is positive or not.

(i) Consider first the situations in which $d \leq p_1$. Clearly, any pricing strategy for firm 1 that involves $p_a > 1 + d + \beta$ is (weakly) dominated by a strategy with $p_a = 1 + d + \beta$. Thus, in what follows we consider strategies such that $p_a \leq 1 + d + \beta$. Similarly, firm 1 cannot be playing any pricing strategy that involves $p_a < 0$ because it is (weakly) dominated by a strategy with $p_a = 0$.

Given that we can restrict attention to $p_a \in [0, 1 + d + \beta]$, we distinguish the following cases that may arise based on the firms' prices, analyzing the implications of each in constructing the firms' demand functions/correspondences:

(a) $U_{b_1} > U_{b_2}$.

(b) $U_{b_1} = U_{b_2}$.

(c) $U_{b_2} > U_{b_1}$.

Regarding case (a), $U_{b_1} > U_{b_2}$ implies that $U_{ab_1}(v) > U_{ab_2}(v)$ for all v (as $d + \beta \geq 0$), which in turn implies that $n_2 = 0$. Therefore, we have that $U_{b_1} > U_{b_2}$ if and only if

$$u + \alpha - p_1 > u - p_2.$$

This shows that $n_2 = 0$ can arise as firm 2's quantity demanded whenever $p_2 > p_1 - \alpha$ is met.

As for case (b), $U_{b_1} = U_{b_2}$ implies that $U_{ab_1}(v) > U_{ab_2}(v)$ for all v whenever $d + \beta > 0$ (we treat the case $d = \beta = 0$ separately further below). Those consumers with valuation greater than $p_a - d - \beta$ purchase both a and b_1, whereas those consumers with valuation smaller than $p_a - d - \beta$ do not purchase a and are indifferent between b_1 and b_2. If $p_a \leq d + \beta$, then $n_2 = 0$ must hold. If instead $p_a \in (d + \beta, 1 + d + \beta]$, then the measure of consumers who purchase b_2 only cannot exceed the total number of consumers who do not purchase a and b_1, which is equal to $p_a - d - \beta$. Hence, $0 \leq n_2 \leq p_a - d - \beta$ must hold, with $U_{b_1} = U_{b_2}$ yielding that it must also hold that

$$p_2 = p_1 + (2n_2 - 1)\alpha.$$

Note that $n_2 = 0$ is also consistent with $p_a > d + \beta$. Then, we have shown that $n_2 = 0$ can arise as firm 2's quantity demanded if $p_2 = p_1 - \alpha$, and

$$n_2 = \frac{p_2 - p_1 + \alpha}{2\alpha} \in (0, p_a - d - \beta]$$

can arise as firm 2's quantity demanded if $p_a \in (d + \beta, 1 + d + \beta]$ and $p_2 = p_1 + (2n_2 - 1)\alpha$, in which case the restrictions on n_2 imply that $p_1 - \alpha < p_2 \leq p_1 - \alpha + 2\alpha(p_a - d - \beta)$.

When $d = \beta = 0$ in case (b), note that $U_{b_1} = U_{b_2}$ implies that $U_{ab_1}(v) = U_{ab_2}(v)$ for all v. Those consumers with valuation greater than p_a purchase a and are indifferent between buying b_1 and b_2 as well, whereas those consumers with valuation smaller than p_a do not purchase a and are indifferent between b_1 and b_2. Clearly, $0 \leq n_2 \leq 1$ must hold, with $U_{b_1} = U_{b_2}$ yielding that it must also hold that

$$p_2 = p_1 + (2n_2 - 1)\alpha,$$

and then

$$n_2 = \frac{p_2 - p_1 + \alpha}{2\alpha}$$

can arise as firm 2's quantity demanded if $-\alpha \leq p_2 \leq \alpha$ whenever $d = \beta = 0$.

Finally, with regards to case (c), $U_{b_2} > U_{b_1}$ implies that $p_2 < p_1 + (2n_2 - 1)\alpha$ must be satisfied. Depending on the comparison between $U_{ab_2}(v)$ and $U_{ab_1}(v)$, we have three subcases:

(c1) $U_{ab_2}(v) > U_{ab_1}(v)$ for all v.

(c2) $U_{ab_2}(v) = U_{ab_1}(v)$ for all v.

(c3) $U_{ab_2}(v) < U_{ab_1}(v)$ for all v.

In case (c1), $U_{b_2} > U_{b_1}$ and $U_{ab_2}(v) > U_{ab_1}(v)$ for all v, which means that no consumer wants to buy b_1, and therefore $n_2 = 1$. Inequality $U_{ab_2}(v) > U_{ab_1}(v)$ implies $p_2 < p_1 - d + (2n_2 - 1)\alpha - \beta$. Note that this condition is consistent with the condition for $U_{b_2} > U_{b_1}$, which is $p_2 < p_1 + (2n_2 - 1)\alpha$. Given that $n_2 = 1$, this case holds if $p_2 < p_1 - d + \alpha - \beta$.

In case (c2), $U_{ab_2}(v) = U_{ab_1}(v)$ implies $p_2 = p_1 + (2n_2 - 1)\alpha - d - \beta$. Then $U_{b_2} > U_{ab_2}(v) = U_{ab_1}(v)$ for consumers whose valuation is weakly smaller than p_a and $U_{ab_2}(v) = U_{ab_1}(v) \geq U_{b_2}$ for consumers whose valuation exceeds p_a. As a result, any consumer with valuation smaller than p_a must purchase b_2, and any consumer with valuation greater than p_a must purchase a and either b_1 or b_2 (whichever of them). The measure of consumers who purchase only b_2 must be greater than or equal to $\min\{1, p_a\}$, so any $n_2 \in [\min\{1, p_a\}, 1]$ can arise as firm 2's quantity demanded if $p_2 = p_1 - d + (2n_2 - 1)\alpha - \beta$ holds. Note that $n_2 = 1$ is consistent with fulfilled expectations if $p_2 = p_1 - d + \alpha - \beta$, and

$$n_2 = \frac{p_2 - p_1 + d + \alpha + \beta}{2\alpha} \in [p_a, 1)$$

is consistent if $p_1 - d + 2\alpha p_a - \alpha - \beta < p_2 < p_1 - d + \alpha - \beta$, and $0 \leq p_a < 1$.

In case (c3), $U_{ab_2}(v) < U_{ab_1}(v)$ implies $p_1 - d + (2n_2 - 1)\alpha - \beta < p_2 < (2n_2 - 1)\alpha$. Let

$$\widehat{v} \equiv p_a - d - \beta + (2n - 1)\alpha + p_1 - p_2$$

denote the unique value of v such that $U_{b_2} = U_{ab_1}(v)$ holds. Note that $\widehat{v} \in (p_a - d - \beta, p_a)$, and as $p_a - d - \beta$ may be negative, we need to consider cases in which $\widehat{v}$ is above 0 and below 0. Similarly, p_a may exceed 1, so we need to consider cases in which $\widehat{v}$ is above 1 and below 1.

Firstly, suppose that $\widehat{v} \leq 0$ (which necessarily requires $p_a < d + \beta$), so that $p_a - d - \beta + (2n - 1)\alpha + p_1 - p_2 \leq 0$. Then, we must have $n_2 = 0$, and the condition becomes $p_1 + p_a - d - \alpha - \beta \leq p_2$, which must hold at the same time as $p_1 - d - \alpha - \beta < p_2 < p_1 - \alpha$. Hence, we simply need that $p_1 + p_a - d - \alpha - \beta \leq p_2 < p_1 - \alpha$.

Secondly, suppose that $\widehat{v} \geq 1$ (which necessarily requires that $p_a > 1$), so that $p_2 \leq p_1 + (2n - 1)\alpha + p_a - d - \beta - 1$. In order for consumers expectations to be fulfilled, it must hold that $n_2 = 1$, so the condition becomes $p_2 \leq p_1 + \alpha + p_a - d - \beta - 1$. Recall that we also need $p_1 + (2n_2 - 1)\alpha - d - \beta < p_2 < p_1 + (2n_2 - 1)\alpha$, but given that we only consider cases such that $p_a \leq 1 + d + \beta$, the final restriction is $p_1 - d + \alpha - \beta < p_2 \leq p_1 + p_a - d - 1 + \alpha - \beta$.

Lastly, suppose now that $\widehat{v} \in (0, 1)$, that is, $p_1 + p_a - d - 1 + (2n_2 - 1)\alpha - \beta < p_2 < p_1 + p_a - d + (2n_2 - 1)\alpha - \beta$. Then n_2 must equal the measure of consumers whose valuation is smaller than $\widehat{v}$ (as $U_{b_2} > U_{ab_1}(v) > U_{ab_2}(v)$ and $U_{b_2} > U_{b_1}$ for such consumers). That is, $n_2 = \Pr(v \leq \widehat{v})$ must hold in order for consumers expectations to be fulfilled (as required by Nash equilibrium behavior), so the fact that $\Pr(v \leq \widehat{v}) = \widehat{v}$ implies that we must have $n_2 = (2n_2 - 1)\alpha + p_a - d - \beta + p_1 - p_2$, which yields

$$n_2 = \frac{p_2 - p_1 - p_a + d + \alpha + \beta}{2\alpha - 1}.$$

The constraints that $p_1 + p_a - d - 1 + (2n_2 - 1)\alpha - \beta < p_2 < p_1 + p_a - d + (2n_2 - 1)\alpha - \beta$ and $p_1 - d + (2n_2 - 1)\alpha - \beta < p_2 < p_1 + \alpha(2n_2 - 1)$ imply that $\max(0, p_a - d - \beta) < n_2 < \min(1, p_a)$.

We must have $p_1 + p_a - d - \alpha - \beta - (1-2\alpha)\max\{p_a - d - \beta, 0\} < p_2 < p_1 + p_a - d - \alpha - \beta - (1-2\alpha)\min\{p_a, 1\}$ when $\alpha > 1/2$.

To summarize, $d \leq p_1$ implies the following:

- $n_2 = 0$ if $p_2 \geq p_1 - \alpha - \max\{d + \beta - p_a, 0\}$. In this case, firm 1 makes sales of good a equal to $1 - (p_a - d - \beta)$, of good b_1 equal to 1, and firm 2 makes no sales.

- $n_2 = 1$ if $p_2 \leq p_1 - d + \alpha - \beta + \max\{p_a - 1, 0\}$. In this case, firm 1 makes sales of good a equal to $1 - p_a$, of good b_1 equal to 0, and firm 2 makes sales of good b_2 equal to 1.

- $n_2 = \dfrac{p_2 - p_1 + \alpha}{2\alpha}$ if $p_1 - \alpha < p_2 \leq p_1 - \alpha + 2\alpha(p_a - d - \beta)$ and $p_a \in (d + \beta, 1 + d + \beta)$. In this case, firm 1 makes sales of good a equal to $1 - (p_a - d - \beta)$, of good b_1 equal to $1 - n_2$, and firm 2 makes sales of good b_2 equal to n_2.

- $n_2 = \dfrac{p_2 - p_1 + d + \alpha + \beta}{2\alpha}$ if $p_1 + 2\alpha p_a - d - \alpha - \beta \leq p_2 < p_1 - d + \alpha - \beta$ and $p_a \in [0, 1)$. In this case, firm 1 makes sales of good a equal to $1 - p_a$, of good b_1 equal to $1 - n_2$, and firm 2 makes sales of good b_2 equal to n_2.

- $n_2 = \dfrac{p_2 - p_1 - p_a + d + \alpha + \beta}{2\alpha - 1}$ if $p_1 + p_a - d - \alpha - \beta + (2\alpha - 1)\max\{p_a - d - \beta, 0\} < p_2 \leq p_1 + p_a - d - \alpha - \beta + (2\alpha - 1)\min\{p_a, 1\}$. In this case, firm 1 makes sales of good a equal to $1 - n_2$, of good b_1 equal to $1 - n_2$, and firm 2 makes sales of good b_2 equal to n_2.

  The graphical representation of firm 2's demand correspondence for both the present and the following case are relatively easy to draw and are not reported as this section is already very long. They are available upon request.

(ii) Consider now the situations in which $d > p_1$. Again, any pricing strategy for firm 1 that involves $p_a > 1 + d + \beta$ is (weakly) dominated by a strategy with $p_a = 1 + d + \beta$. Thus, in what follows we consider strategies such that $p_a \leq 1 + d + \beta$. Similarly, firm 1 cannot be playing any pricing strategy that involves $p_a < 0$ because it is (weakly) dominated by a strategy with $p_a = 0$. Finally, note when $d > p_1$ that we cannot have $p_1 < 0$, because in such a case a consumer buying b_1 would be subsidized by firm 1 without firm 1 benefiting from it. Moreover, we need to assume that $p_a + p_1 - d \geq 0$, *i.e.* the sum of prices charged by firm 1 cannot be negative.

Given that we can restrict attention to $p_a \in [0, 1 + d + \beta]$, we distinguish the following cases that may arise based on the firms' pricing, analyzing the implications of each in constructing the firms' demand functions/correspondences:

(a) $U_{b_1} > U_{b_2}$.

(b) $U_{b_1} = U_{b_2}$.

(c) $U_{b_2} > U_{b_1}$.

Regarding case (a), $U_{b_1} > U_{b_2}$ implies that $U_{ab_1}(v) > U_{ab_1b_2}(v)$ for all v (as $p_1 + \beta \geq 0$), which in turn implies that $n_2 = 0$. Therefore, we have that $U_{b_1} > U_{b_2}$ if and only if

$$u + \alpha - p_1 > u - p_2.$$

This shows that $n_2 = 0$ can arise as firm 2's quantity demanded whenever $p_2 > p_1 - \alpha$ is met.

As for case (b), $U_{b_1} = U_{b_2}$ implies that $U_{ab_1}(v) > U_{ab_1b_2}(v)$ for all v whenever $p_1 > 0$ (we treat the case $p_1 = \beta = 0$ separately further below). Those consumers with valuation greater than $p_a - d - \beta$ purchase both a and b_1, whereas those consumers with valuation smaller than $p_a - d - \beta$ do not purchase a and are indifferent between b_1 and b_2. If $p_a \leq d + \beta$, then $n_2 = 0$ must hold. If instead $p_a \in (d + \beta, 1 + d + \beta]$, then the measure of consumers who purchase b_2 only cannot exceed the total number of consumers who do not purchase a and b_1, which is equal to $p_a - d - \beta$. Hence, $0 \leq n_2 \leq p_a - d - \beta$ must hold, with $U_{b_1} = U_{b_2}$ yielding that it must also hold that

$$p_2 - p_1 = (2n_2 - 1)\alpha.$$

Note that $n_2 = 0$ is also consistent with $p_a > d + \beta$. Then, we have shown that $n_2 = 0$ can arise as firm 2's quantity demanded if $p_2 = p_1 - \alpha$, and

$$n_2 = \frac{p_2 - p_1 + \alpha}{2\alpha} \in (0, p_a - d - \beta]$$

can arise as firm 2's quantity demanded if $p_a > d + \beta$ and $p_2 = p_1 + (2n_2 - 1)\alpha$, in which case the restrictions on n_2 imply that $p_1 - \alpha < p_2 \leq p_1 - \alpha + 2\alpha(p_a - d - \beta)$.

When $p_1 = \beta = 0$ in case (b), note that $U_{b_1} = U_{b_2}$ implies that $U_{ab_1}(v) = U_{ab_1b_2}(v)$ for all v. Those consumers with valuation greater than $p_a - d$ purchase a and are indifferent between buying b_1 and b_2 as well, whereas those consumers with valuation smaller than $p_a - d - \beta$ do not purchase a and are indifferent between b_1 and b_2. Clearly, $0 \leq n_2 \leq 1$ must hold, with $U_{b_1} = U_{b_2}$ yielding that it must also hold that

$$p_2 = (2n_2 - 1)\alpha,$$

so

$$n_2 = \frac{p_2 + \alpha}{2\alpha}$$

can arise as firm 2's quantity demanded if $-\alpha \leq p_2 \leq \alpha$ whenever $p_1 = \beta = 0$.

Finally, with regards to case (c), $U_{b_2} > U_{b_1}$ implies that $p_2 < p_1 + (2n_2 - 1)\alpha$ must be satisfied. Depending on the comparison between $U_{ab_2}(v)$ and $U_{ab_1}(v)$, we have three subcases:

(c1) $U_{ab_1b_2}(v) > U_{ab_1}(v)$ for all v.

(c2) $U_{ab_1b_2}(v) = U_{ab_1}(v)$ for all v.

(c3) $U_{ab_1b_2}(v) < U_{ab_1}(v)$ for all v.

Note that, when $p_1 = \beta = 0$, only case (c1) is possible, whereas all three cases are possible when $p_1 + \beta > 0$.

In case (c1), $U_{b_2} > U_{b_1}$ and $U_{ab_1b_2}(v) > U_{ab_1}(v)$ for all v, which means that no consumer wants to consume b_1 even if they buy it, and therefore $n_2 = 1$. Inequality $U_{ab_1b_2}(v) > U_{ab_1}(v)$ implies $p_2 < (2n_2 - 1)\alpha - \beta$. Note that this condition is consistent with the condition for $U_{b_2} > U_{b_1}$, which is $p_2 < p_1 + (2n_2 - 1)\alpha$. Given that $n_2 = 1$, this case holds if $p_2 < \alpha - \beta$. Note that those consumers whose valuation exceeds $p_a + p_1 - d$ purchase all goods, whereas those consumers whose valuation is below $p_a + p_1 - d$ simply purchase b_2.

In case (c2), $U_{ab_1b_2}(v) = U_{ab_1}(v)$ implies $p_2 = (2n_2 - 1)\alpha - \beta$. Take into account that $U_{b_2} > U_{b_1}$, so any consumer with valuation smaller than $p_a + p_1 - d$ must purchase b_2 only, whereas any consumer with valuation greater than $p_a + p_1 - d$ must purchase firm 1's bundle (perhaps together with b_2). Therefore, the measure of consumers who purchase only b_2 must be greater than or equal to $\min\{1, p_a + p_1 - d\}$, so any $n_2 \in [\min\{1, p_a + p_1 - d\}, 1]$ can arise as firm 2's quantity demanded if $p_2 = (2n_2 - 1)\alpha - \beta$ holds. Note that $n_2 = 1$ is consistent with fulfilled expectations if $p_2 = \alpha - \beta$, and

$$n_2 = \frac{p_2 + \alpha + \beta}{2\alpha} \in [p_a + p_1 - d, 1)$$

is consistent if $-\alpha - \beta + 2\alpha(p_a + p_1 - d) < p_2 < \alpha - \beta$, and $0 \leq p_a + p_1 - d < 1$, as we previously specified.

In case (c3), $U_{ab_1b_2}(v) < U_{ab_1}(v)$ implies $(2n_2 - 1)\alpha - \beta < p_2 < (2n_2 - 1)\alpha + p_1$. Let

$$\widehat{v} \equiv p_a - d - \beta + (2n - 1)\alpha + p_1 - p_2$$

denote the unique value of v such that $U_{b_2} = U_{ab_1}(v)$ holds. Note that $\widehat{v} \in (p_a - d - \beta, p_a + p_1 - d)$, and as $p_a - d - \beta$ may be negative, we need to consider cases in which $\widehat{v}$ is above 0 and below 0. Similarly, $p_a + p_1 - d$ may exceed 1, so we need to consider cases in which $\widehat{v}$ is above 1 and below 1.

Firstly, suppose that $\widehat{v} \leq 0$ (which necessarily requires $p_a < d + \beta$), so that $p_a - d - \beta + (2n - 1)\alpha + p_1 - p_2 \leq 0$. Then, we must have $n_2 = 0$, so the condition becomes $p_1 + p_a - d - \beta - \alpha \leq p_2$, which must hold at the same time as $-\alpha - \beta < p_2 < p_1 - \alpha$, so we simply need that $p_1 + p_a - d - \beta - \alpha \leq p_2 < p_1 - \alpha$.

Secondly, suppose that $\widehat{v} \geq 1$ (which necessarily requires that $p_a + p_1 - d > 1$), so that $p_2 \leq p_1 + p_a - d - 1 + (2n - 1)\alpha - \beta$. In order for consumers expectations to be fulfilled, it must hold that $n_2 = 1$, so the condition becomes $p_2 \leq p_1 + p_a - d - 1 + \alpha - \beta$. Recall we also need $(2n_2 - 1)\alpha - \beta < p_2 < p_1 + (2n_2 - 1)\alpha$, but given that we only consider cases such that $p_a \leq 1 + d + \beta$, the final restriction is $\alpha - \beta < p_2 \leq p_1 + p_a - d - 1 + \alpha - \beta$. Note that $U_{b_2} > U_{ab_1b_2}(v)$ for all $v \in [0, 1]$ implies that $p_a + p_1 - d - \beta > 1$, so firm 1 does no sell any unit of good a.

Lastly, suppose now that $\widehat{v} \in (0, 1)$, that is, $p_1 + p_a - d - 1 + (2n - 1)\alpha - \beta < p_2 < p_1 + p_a - d + (2n - 1)\alpha - \beta$. Then n_2 must equal the measure of consumers whose valuation is smaller than $\widehat{v}$ (because $U_{b_2} > U_{ab_1}(v) > U_{ab_1b_2}(v)$ and $U_{b_2} > U_{b_1}$ for such consumers). That is, $n_2 = \Pr(v \leq \widehat{v})$ must hold in order for consumers expectations to be fulfilled (as required by Nash equilibrium behavior), so the fact that $\Pr(v \leq \widehat{v}) = \widehat{v}$ implies that we must have $n = (2n-1)\alpha + p_a - d - \beta + p_1 - p_2$, which yields

$$n_2 = \frac{p_2 - p_1 - p_a + d + \alpha + \beta}{2\alpha - 1}.$$

The constraints that $p_1 + p_a - d - 1 + (2n-1)\alpha - \beta < p_2 < p_1 + p_a - d + (2n-1)\alpha - \beta$ and $(2n-1)\alpha - \beta < p_2 < p_1 + \alpha(2n-1)$ imply that $\max(0, p_a - d - \beta) < n_2 < \min(1, p_a + p_1 - d)$. We must have $p_1 + p_a - d - \beta - \alpha + (2\alpha - 1)\max\{p_a - d - \beta, 0\} < p_2 < p_1 + p_a - d - 1 - \beta - \alpha + (2\alpha - 1)\min\{p_a + p_1 - d - 1, 0\}$ when $\alpha > 1/2$.

We can now summarize the situations in which $d > p_1$:

- $n_2 = 0$ if $p_2 \geq p_1 - \alpha - \max\{d + \beta - p_a, 0\}$. In this case, firm 1 makes sales of good a equal to $1 - (p_a - d - \beta)$, of good b_1 equal to 1, and firm 2 makes no sales.

- $n_2 = 1$ if $p_2 \leq \alpha - \beta + \max\{p_1 + p_a - d - 1, 0\}$. In this case, firm 1 makes sales of good a equal to $1 - (p_a + p_1 - d)$, of good b_1 equal to $1 - (p_a + p_1 - d)$, and firm 2 makes sales of good b_2 equal to 1.

- $n_2 = \dfrac{p_2 - p_1 + \alpha}{2\alpha}$ if $p_1 - \alpha < p_2 \leq p_1 - \alpha + 2\alpha(p_a - d - \beta)$ and $p_a \in (d + \beta, 1 + d + \beta)$. In this case, firm 1 makes sales of good a equal to $1 - (p_a - d - \beta)$, of good b_1 equal to $1 - n_2$, and firm 2 makes sales of good b_2 equal to n_2.

- $n_2 = \dfrac{p_2 + \alpha + \beta}{2\alpha}$ if $2\alpha(p_a + p_1 - d) - \alpha - \beta \leq p_2 < \alpha - \beta$, and $0 \leq p_1 + p_a - d < 1$. In this case, firm 1 makes sales of good a equal to $1 - (p_a + p_1 - d)$, of good b_1 equal to $1 - n_2$, and firm 2 makes sales of good b_2 equal to n_2.

- $n_2 = \dfrac{p_2 - p_1 - p_a + d + \alpha + \beta}{2\alpha - 1}$ if $p_1 + p_a - d - \alpha - \beta + (2\alpha - 1)\max\{p_a - d - \beta, 0\} < p_2 < p_1 + p_a - d - \beta - \alpha - 1 + (2\alpha - 1)\min\{p_a + p_1 - d - 1, 0\}$. In this case, firm 1 makes sales of good a equal to $1 - n_2$, of good b_1 equal to $1 - n_2$, and firm 2 makes sales of good b_2 equal to n_2.

We now consider the equilibrium analysis, which requires two steps. Denote equilibrium values using the "MS" superscript. We first demonstrate that the following lemma holds.

Lemma 9 (Mixed bundling with a specific complementarity). *An equilibrium exists. Equilibria are such that:*

(a) When $\alpha < \beta + c$ and $\alpha \leq 3/4 + \beta$, $p_2^{MS} = c$ and any $(p_1^{MS}, p_a^{MS}, d^{MS})$ such that $p_1^{MS} + p_a^{MS} - d^{MS} = 1 + \beta - \alpha$, with $1 \leq p_a^{MS} \leq d^{MS} + \beta$ and $d^{MS} \geq 1 - \beta$, is an equilibrium; in any of such equilibria, $n_2^{MS} = 0$, $\pi_1^{MS} = 1 + \beta - \alpha > 0$ and $\pi_2^{MS} = 0$.

(b1) When $\alpha \geq \beta + c$ and $c \leq (2 + \beta^2)/4$, $p_2^{MS} = \alpha - \beta$ and any $(p_1^{MS}, p_a^{MS}, d^{MS})$ such that $p_a^{MS} = 1/2$, with $p_1^{MS} - d^{MS} = 0$, is an equilibrium; in any of such equilibria, $n_2^{MS} = 1$, no consumer buys good b_1, $\pi_1^{MS} = 1/4$ and $\pi_2^{MS} = \alpha - \beta - c$.

(b2) When $c > (2 + \beta^2)/4$ and $\alpha \geq \beta + (2 + \beta^2)/4$, $p_2^{MS} = c + \alpha - [\beta + (2 + \beta^2)/4]$ and any $(p_1^{MS}, p_a^{MS}, d^{MS})$ such that $p_a^{MS} = 1/2$ with $p_1^{MS} - d^{MS} = c - (2 + \beta^2)/4$ is an

equilibrium; in any of such equilibria, $n_2^{MS} = 1$*, no consumer buys good* b_1*,* $\pi_1^{MS} = 1/4$ *and* $\pi_2^{MS} = \alpha - [\beta + (2 + \beta^2)/4]$.

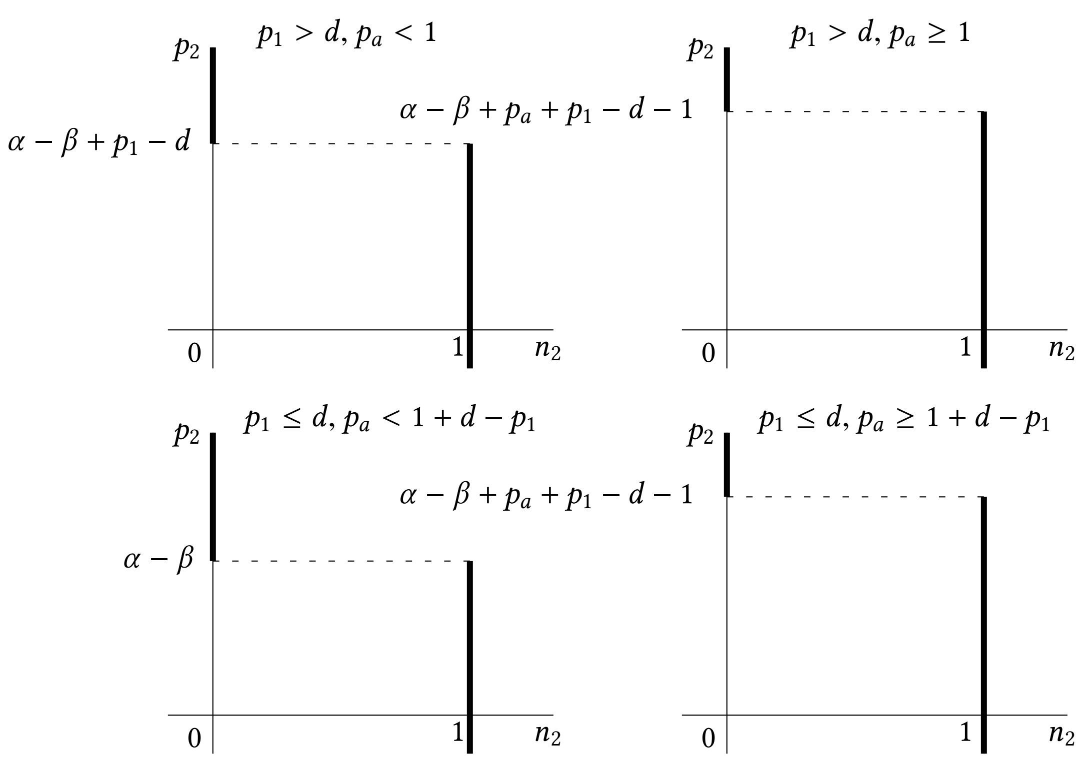
Proof. In our equilibrium analysis, we will make use of Figure C, which represents firm 2's demand function based on our previous analysis. Let us first try to sustain $n_2 = 1$ in equilibrium with no sales of good b_1. Clearly, in an equilibrium with no sales of good b_1, it must hold that $p_a^{MS} = 1/2$ and $p_2^{MS} = \alpha - \beta + \max\{0, p_1^{MS} - d^{MS}\} \geq c$ (see left panels in Figure C).

Case (i): Suppose first that $p_1^{MS} - d^{MS} > 0$. If firm 1 deviates by charging $p_a < 1$ and $p_1 > d$, then the best it can do is to charge $p_1 = p_2^{MS} + \beta - \alpha + d - \epsilon = p_1^{MS} + d - d^{MS} - \epsilon > d$ (see Figure C, top panel on the left) so as to earn

$$p_1^{MS} + d - d^{MS} + (p_a - d)[\min\{1, 1 + d + \beta - p_a\}] =$$
$$= \begin{cases} p_1^{MS} - d^{MS} + p_a & \text{if } p_a \leq d + \beta \\ p_1^{MS} + d - d^{MS} + (p_a - d)(1 + d + \beta - p_a) & \text{if } p_a > d + \beta \end{cases}$$

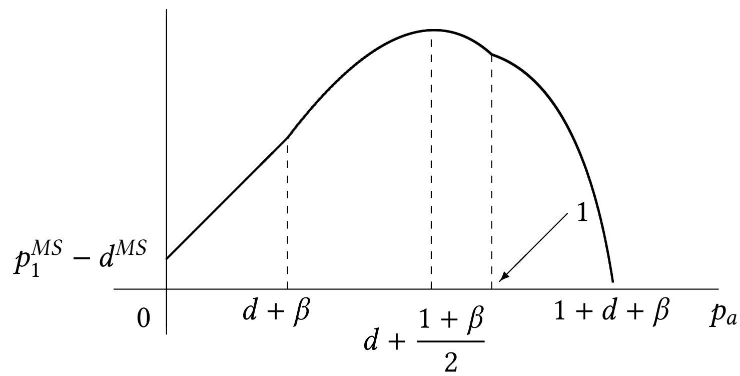
If instead firm 1 chooses $p_a \geq 1$ and $p_1 > d$ when deviating, clearly the best it can do is to charge $p_1 = p_2^{MS} + \beta - \alpha + d + 1 - p_a - \epsilon = p_1^{MS} + d - d^{MS} - (p_a - 1) - \epsilon$ (see Figure C, top panel on the right) so as to earn $p_1^{MS} + d - d^{MS} - c + 1 - p_a + (p_a - d)[\min\{1, 1 + d + \beta - p_a\}]$. Note that $p_1 > d$ if and only if $p_a < 1 + p_1^{MS} - d^{MS}$, which implies that, as soon as $p_a > 1 + p_1^{MS} - d^{MS}$ (and so $p_1 < d$), it holds that $p_1 + p_a - d = p_1^{MS} - d^{MS} + 1 - \epsilon > 1$ (see Figure C, bottom panel on the right), so that firm 1 continues to earn $p_1^{MS} + d - d^{MS} - c + 1 - p_a + (p_a - d)[\min\{1, 1 + d + \beta - p_a\}]$.

Figure C



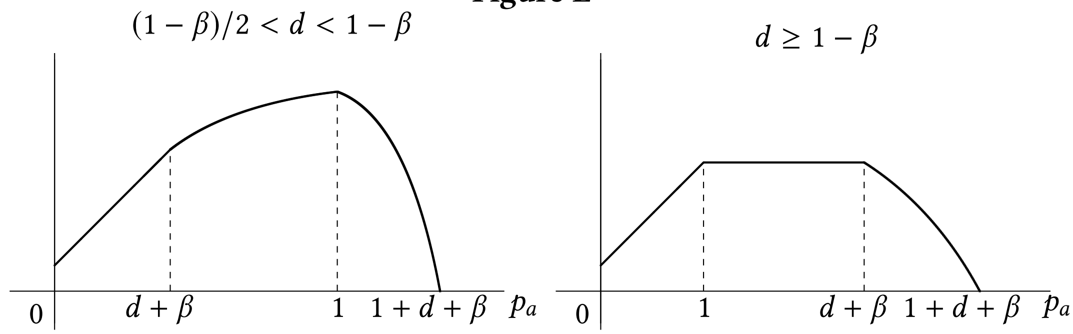
Recalling that $\beta < 1$, if $d \le (1-\beta)/2$, then the payoff to deviating as a function of p_a is represented in Figure D:

Figure D



As a consequence, when $d \le (1-\beta)/2$, the optimal deviation (taking d as exogenous) is given by $\widehat{p_1} = d + p_1^{MS} - d^{MS} - \epsilon > d$ and $\widehat{p_a} = d + (1+\beta)/2$ so as to earn $p_1^{MS} - d^{MS} + d - c + (1+\beta)^2/4$. The following figure represents the payoff to deviating as a function of p_a when $(1-\beta)/2 < d < 1-\beta$ and $d \ge 1-\beta$:

Figure E



Hence, when $(1-\beta)/2 < d$, the optimal deviation (taking d as exogenous) is given by $\widehat{p_1} = d + p_1^{MS} - d^{MS} - \epsilon > d$ and $\widehat{p_a} = 1$ so as to earn

$$p_1^{MS} - d^{MS} + d - c + (1-d)\min\{1, d+\beta\}.$$

Because $(1+\beta)^2/4 > (1-d)\min\{1, d+\beta\}$ for $d > (1-\beta)/2$, when maximizing firm 1's deviation profit with respect to d, we find that the optimal deviation involves $\widehat{d} = (1-\beta)/2$, and hence firm 1 can earn $p_1^{MS} - d^{MS} + (1-\beta)/2 - c + (1+\beta)^2/4$. Such a deviation is unprofitable if and only if $1/4 \ge p_1^{MS} - d^{MS} + (1-\beta)/2 - c + (1+\beta)^2/4$, that is, $c - (2+\beta^2)/4 \ge p_1^{MS} - d^{MS}$.

Case (ii): Suppose now that $p_1^{MS} - d^{MS} \le 0$. Clearly, the only reason for firm 1 to deviate is to attract consumers in market B. Because $p_2^{MS} = \alpha - \beta$, it is clearly impossible for firm 1 to find $p_1 > d$ such that it ends up attracting demand in market B (see top panels of Figure C). Similarly, the fact that $p_2^{MS} = \alpha - \beta$ implies that it is impossible to find $p_1 \le d$ and $p_a \ge 1 + d - p_1$ such that it ends up attracting demand in market B (see right bottom panel of Figure C). As a result, firm 1 has no profitable deviation available, so existence of an equilibrium with $n_2 = 1$ and no sales of good b_1 would simply require $\alpha \geq \beta + c$.

Note that ruling out weakly dominated strategies implies that $p_1^{MS} - d^{MS}$ must be as high as admissible based on our analysis of cases (i) and (ii). If $c \leq (2+\beta^2)/4$, then the conditions required for case (i) (namely, $p_1^{MS} - d^{MS} > 0$ and $p_1^{MS} - d^{MS} \leq c - (2+\beta^2)/4$) cannot possibly hold, so case (ii) implies that an equilibrium with $n_2 = 1$ and no sales of good b_1 exists if $\alpha \geq \beta + c$. If instead $c > (2+\beta^2)/4$, it must hold that $p_1^{MS} - d^{MS} = c - (2+\beta^2)/4 > 0$, so case (i) implies that that an equilibrium with $n_2^{MS} = 1$ and no sales of good b_1 exists if $\alpha \geq \beta + (\beta^2+2)/4$ (so that firm 2 has no incentive to deviate because $p_2^{MS} = \alpha - \beta + p_1^{MS} - d^{MS} \geq c$).

We have therefore shown that $c \leq (2+\beta^2)/4$ and $\alpha \geq \beta + c$ imply that the following equilibrium with $n_2^{MS} = 1$ and no sales of good b_1 exists: $p_a^{MS} = 1/2$, $p_1^{MS} - d^{MS} = 0$, and $p_2^{MS} = \alpha - \beta \geq c$, with $\pi_1^{MS} = 1/4$ and $\pi_2^{MS} = \alpha - \beta - c$. In turn, $c > (2+\beta^2)/4$ and $\alpha \geq \beta + (\beta^2+2)/4$ imply that the following equilibrium with $n_2^{MS} = 1$ and no sales of good b_1 exists: $p_a^{MS} = 1/2$, $p_1^{MS} - d^{MS} = c - (2+\beta^2)/4 > 0$, and $p_2^{MS} = c + \alpha - [\beta + (2+\beta^2)/4] \geq c$, with $\pi_1^{MS} = 1/4$ and $\pi_2^{MS} = \alpha - [\beta + (2+\beta^2)/4]$.

Let us now try to sustain $n_2 = 1$ in equilibrium with positive sales of good b_1 (but no usage of such a good). In such a case, we must have $p_a^{MS} + p_1^{MS} - d^{MS} = \min\{1, (1+c)/2\} < 1$ and $p_2^{MS} = \alpha - \beta$, with $p_1^{MS} - d^{MS} \leq 0$ (so $p_a^{MS} \geq (1-c)/2$). Clearly, we need $c < \alpha - \beta$ and $\alpha > \beta$. We also need that firm 1 has no profitable unilateral deviation. Note that it is impossible for firm 1 to profitably deviate by attempting to conquer market B through some $p_1 > d$ (perhaps changing p_a at the same time): otherwise, such a deviation should be accompanied by price $p_1 = p_2^{MS} + d + \beta - \alpha - \max\{0, p_a - 1\} - \epsilon = d - \max\{0, p_a - 1\} - \epsilon$ charged for b_1, which contradicts $p_1 > d$. So consider deviations such that $p_1 \leq d$ from now on. In such a case, firm 1 cannot find any price such that firm 2 loses its demand, since $p_2^{MS} = \alpha - \beta$ guarantees that firm 2 sells to all consumers in market B regardless of whether firm 1 charges prices such that $p_1 + p_a - d$ exceeds 1 or not (see Figure C, bottom panels). However, firm 1 can always neglect market B and focus on monopolizing market A so as to earn 1/4. Because $1/4 > (\max\{0, (1-c)/2\})^2$, there cannot exist an equilibrium with $n = 1$ and positive sales of good b_1 (but no usage of such a good).

We conclude by trying to sustain $n_2 = 0$ in equilibrium, noting that it must clearly hold that $p_2^{MS} = c$ in such an equilibrium (if it exists). Consider first the cases in which $c \leq \alpha - \beta$. For fixed d and p_a, suppose that firm 1 chooses to price according to $p_1 = p_2^{MS} + d + \beta - \alpha - \max\{0, p_a - 1\} - \epsilon$ (see top panels in Figure C, as well as the bottom panel on the right). Because $c \leq \alpha - \beta$, $p_1 - d = c + \beta - \alpha - \max\{0, p_a - 1\} - \epsilon < 0$, so we would need $p_a \geq 1 + d - p_1$ (see Figure C's bottom panel on the right). Because $p_1 = c + d + \beta - \alpha - \max\{0, p_a - 1\} - \epsilon$ and $c \leq \alpha - \beta$, it cannot possibly hold that $p_a \geq 1 + d - p_1$, so there cannot exist an equilibrium with $n = 0$ whenever $c \leq \alpha - \beta$.[13]

In sustaining $n_2 = 0$ as an equilibrium outcome, we consider henceforth the cases in which $c > \alpha - \beta$. Clearly, there cannot exist any incentive by firm 2 to unilaterally deviate

[13] Another way to derive this result is to observe in Figure C that, regardless of firm 1's pricing, firm 2 can always charge $p_2 = \alpha - \beta$ and ensure some profit unless $\alpha - \beta < c$.

from $p_2^{MS} = c$ because $c > \alpha - \beta$ and attracting all demand would require lowering the price.

Next, note that $p_1 - d \leq -\max\{0, p_a - 1\}$ ensures that firm 1 gets all the demand in market B (see Figure C's bottom panel on the left), so that firm 1 earns

$$\pi_1(p_1, p_a, d) = p_1 - c + (p_a - d)[\min\{1, 1 + d + \beta - p_a\}].$$

Maximizing $\pi_1(p_1, p_a, d)$ with respect to p_1, p_a and d subject to the constraints that $p_1 \leq d$ if $p_a < 1$ or $p_1 - d \leq 1 - p_a$ if $p_a \geq 1$ yields that firm 1 earns a profit equal to $1 - c$. Proving this is quite elaborate, so we proceed in steps (roughly speaking, the idea is to successively maximize with respect to p_1, p_a and d in order to take carefully into account important nondifferentiability points of the profit function). Suppose first that $p_a < 1$. Keeping p_a and d fixed, $p_1 = d$ maximizes $\pi_1(p_1, p_a, d)$ with respect to p_1, so firm 1 must maximize $d - c + (p_a - d)[\min\{1, 1 + d + \beta - p_a\}]$ with respect to d and p_a. Keeping d fixed, it holds because $\beta < 1$ that firm 1's profit is maximized by

$$p_a^{MS} = \begin{cases} d + (1+\beta)/2 & \text{if } d \leq (1-\beta)/2 \\ 1 & \text{if } d > (1-\beta)/2 \end{cases}.$$

It follows that firm 1's profit equals $d + (1+\beta)^2/4$ if $d \leq (1-\beta)/2$ and $d + (1-d)[\min\{1, d + \beta\}]$ if $d \geq (1-\beta)/2$, so any $d^{MS} \geq 1 - \beta$ maximizes firm 1's profit, which is equal to $1 - c$.

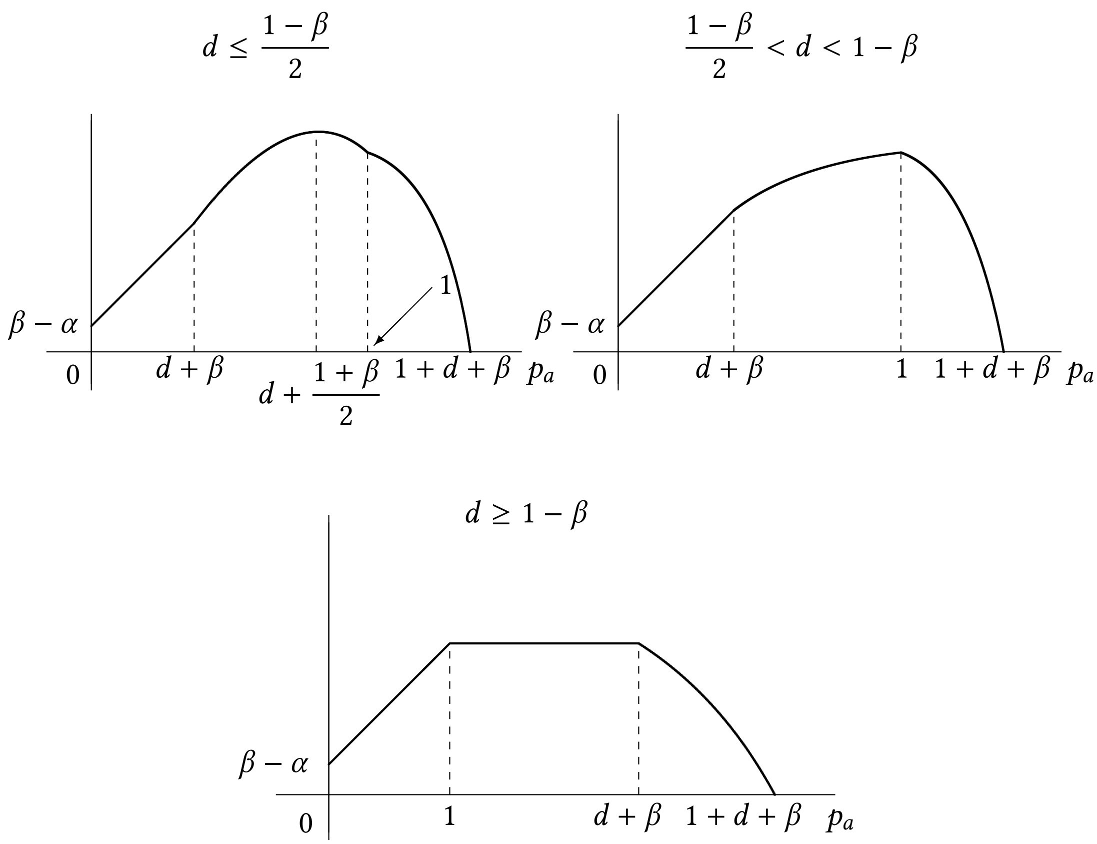
Thus far, we have shown that any $d^{MS} \geq 1 - \beta$ maximizes firm 1's profit when $p_a < 1$, and the maximal profit equals $1 - c$. We now turn to the cases in which $p_a \geq 1$, so that $p_1 = 1 + d - p_a$ maximizes $\Pi_1(p_1, p_a, d)$ with respect to p_1. Then firm 1 must maximize $1 + d - p_a - c + (p_a - d)[\min\{1, 1 + d + \beta - p_a\}]$ with respect to d and p_a. Keeping d fixed, it holds because $\beta < 1$ that firm 1's profit is maximized by $p_a^{MS} = 1$ if $d \leq 1 - \beta$ and any $p_a^{MS} \in [1, d + \beta]$ if $d > 1 - \beta$. It follows that firm 1's profit as a function of d equals $d + (1 - d)\min\{1, d + \beta\}$, which is maximized for any $d^{MS} \geq 1 - \beta$. This shows that any $d^{MS} \geq 1 - \beta$ maximizes firm 1's profit when $p_a \geq 1$, and the maximal profit equals $1 - c$.

Having shown that firm 1 can earn $1 - c$ by (optimally) following a pricing strategy such that $p_1 - d \leq -\max\{0, p_a - 1\}$, we now show that it can earn more than $1 - c$ by (optimally) following a pricing strategy such that $p_1 - d > -\max\{0, p_a - 1\}$. To this end, suppose now that firm 1 follows a pricing strategy such that $p_1 - d > -\max\{0, p_a - 1\}$ (see all the panels in Figure C except the bottom left one). Given p_a and d, firm 1 must be choosing p_1 so that $p_1 = p_2^{MS} + \beta - \alpha + d - \max\{0, p_a - 1\} - \epsilon$ holds, thus earning

$$\pi_1(p_a, d) = p_2^{MS} + d + \beta - \alpha - \max\{0, p_a - 1\} - c + (p_a - d)[\min\{1, 1 + d + \beta - p_a\}].$$

Note that $p_1 - d > -\max\{0, p_a - 1\}$ is directly satisfied when $p_1 = p_2^{MS} + d + \beta - \alpha - \max\{0, p_a - 1\} - \epsilon$ (for $\epsilon > 0$ small enough) because $c > \alpha - \beta$. Firm 1 then maximizes $\pi_1(p_a, d)$ with respect to p_a and d. To do so, we will first maximize $\pi_1(p_a, d)$ with respect to p_a keeping d fixed, find out the optimal value/s of p_a as a function of d, plug them into the profit function, and then maximize the resulting objective function with respect to d. Note that $\Pi_1(p_a, d)$ as a function of p_a is as represented in Figure F:

Figure F



Therefore, $\beta < 1$ implies that $\pi_1(p_a, d)$ is maximized with respect to p_a as follows: $p_a^{MS} = d + (1+\beta)/2$ if $d \leq (1-\beta)/2$, $p_a^{MS} = 1$ if $(1-\beta)/2 \leq d \leq 1-\beta$ and $p_a^{MS} \in [1, d+\beta]$ if $d \geq 1-\beta$. Firm 1's profit as a function of d is as follows:

$$\widehat{\pi}_1(d) = \begin{cases} \beta - \alpha + d + (1+\beta)^2/4 & \text{if } d \leq (1-\beta)/2 \\ \beta - \alpha + d + (1-d)(d+\beta) & \text{if } (1-\beta)/2 \leq d \leq 1-\beta \\ \beta - \alpha + 1 & \text{if } d \geq 1-\beta \end{cases}.$$

This function is clearly maximized for any $d^{MS} \geq 1-\beta$, since it is increasing. Therefore, any triplet $(p_1^{MS}, p_a^{MS}, d^{MS})$ such that $p_1^{MS} + p_a^{MS} - d^{MS} = \beta - \alpha + 1$, $1 \leq p_a^{MS} \leq d^{MS} + \beta$ and $d^{MS} \geq 1-\beta$ maximizes firm 1's profit, so that firm 1 earns $\beta - \alpha + 1$. Clearly, $c > \alpha - \beta$ implies that this profit exceeds the one that firm 1 could earn by setting prices optimally under the constraint that $p_1 - d \leq -\max\{0, p_a - 1\}$ (such a profit was equal to $1-c$). Having shown that firm 1 has no incentive to deviate from this pricing if it is to make sales of good b_1, we need to rule out incentives of such a firm to disregard market B and simply focus on monopolizing market A: in order for $\beta - \alpha + 1 \geq 1/4$, it is necessary that $\alpha \leq \beta + 3/4$ holds. ■

Part (a) of Lemma 9 shows that firm 1 conquers market B by inducing pure bundling of the two goods it sells through the three prices it charges, as $p_a^{MS} \geq 1$. Even though the pricing when no bundling is pursued can be replicated under mixed bundling (by setting the discount equal to 0), firm 1 cannot commit to not offering a discount, and it always has an incentive to use such a lever when competing for market B. So having more degrees of freedom under mixed bundling turns firm 1 a softer competitor for market B and allows firm 2 to defend such a market more easily.

In order to complete our formal demonstration, we need a second step. In particular, consider the private incentive for firm 1 to adopt pure bundling when complementarity is specific. Indeed, because $(\max\{0, (1-c)/2\})^2 \leq 1/4$, $1+\beta-\alpha > \beta-\alpha+(1+\beta)^2/4$, and $1+\beta-\alpha \geq 1/4$ if and only if $\alpha \leq 3/4+\beta$, by comparing Lemma 1 with Lemma 3 we obtain that pure bundling is preferred by firm 1 over unbundled pricing if and only if $\alpha \leq \min\{3/4, c\}+\beta$.

The following proposition comes from combining lemma 9 with the finding that firm 1 prefers pure bundling over unbundled pricing if and only if β is large enough.

Proposition 3 (Mixed bundling with a specific complementarity)**.** *An equilibrium exists. Any equilibrium with mixed bundling yields the same profit or less for firm 1 than the (unique) equilibrium of a game in which firm 1 chooses between unbundled pricing and pure bundling before competing with firm 2.*

In principle, mixed bundling offers firm 1 a much wider choice of pricing strategies. For example, unbundled pricing arises if $d = 0$, whereas pure bundling arises for instance if $p_a = 1$ and $d > 0$ (the only reason why good a would be purchased is because b_1 is purchased as well). Despite the rich pricing afforded by mixed bundling, Proposition 3 reveals that, if firm 1 can choose between unbundled pricing and pure bundling, adding the possibility of choosing mixed bundling before competing with firm 2 does not bring extra profit to firm 1. Thus, there is no loss of generality in restricting our analysis to firm 1's decision between unbundled pricing and pure bundling.